23 April 2012

Conventional Mortgages
Contract interest rate on 30-year fixed rate first mortgages
This week we will talk about various aspects of interest rates, including compound interest, loans, inflation, and credit cards.
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The underlying mathematics of compound interest is the same as what we will need to talk about exponential growth and decay next week. That topic comes up when discussing population growth, radioactive decay, and why you can zoom in on a location in google maps quickly.
With simple interest if you put $100 in a bank account at 5% interest per year, you will have $105 after one year. You earn $5 in interest.
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You can compute this by multiplying the amount of money in the bank by the interest rate, after converting the rate to a decimal. You do that by dividing the rate by 100. So, 5% = 0.05. Then the interest is $100 \cdot 0.05 = $5.00.
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Simple Interest

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If you earn simple interest, then as long as you leave the $100 in the bank you’ll get $5 in interest each year.

While banks do not pay interest in this way, if you withdrew the interest each time then effectively you would be getting simple interest.
Compound interest is how banks actually pay interest. What this means is they pay interest based on the amount of money in the account, regardless of how much was the original principal and how much is interest.
Compounding

Each interest bearing account compounds interest after a certain time interval. For example, the savings account Bank of America is advertising in the graphic below compounds interest each day.
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<table>
<thead>
<tr>
<th>Regular Savings Account Interest Rates</th>
<th>Interest Rate</th>
<th>APY *</th>
</tr>
</thead>
<tbody>
<tr>
<td>All Balances</td>
<td>0.05%</td>
<td>0.05%</td>
</tr>
</tbody>
</table>

Open a savings account now

*Annual Percentage Yield (APY) is accurate as of 04/21/2012. Rates may change at any time without prior notice, before or after the account is opened. Fees could reduce earnings on the account. Minimum opening balance is $25.
To make things simpler, we’ll start talking about an account which compounds each year.
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If you put $100 in a bank account at 5% interest per year, compounded yearly, you will have $105 after one year. You earn $5 in interest.
If you leave the account untouched, how much will you have after 2 years?

A $110

B More than $110

C Less than $110
You’ll have more than $110. Because the bank pays you 5% interest, you get 5% of your balance of $105. The interest you earn in year 2 is then

\[ 105 \cdot 0.05 = 5.25 \]
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$$105 \cdot 0.05 = 5.25$$

So, you earn $5 interest in the first year and $5.25 in the second year, so you’ll have $110.25 at the end of the second year.
After one year, the money in the bank is (principal plus interest)

\[ \$100 + \$100 \cdot 0.05 = \$100 \cdot (1 + 0.05) = \$100 \cdot 1.05 = \$105 \]
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Putting these together, the money in the bank after two years is

\[ \$105 \cdot 1.05 = (\$100 \cdot 1.05) \cdot 1.05 = \$100 \cdot (1.05)^2 = \$110.25 \]
Using exponents is a short way of writing repeated multiplication. That is, $(1.05)^2$ means multiply $1.05$ times $1.05$. Similarly, $(1.05)^3$ means multiply $1.05 \cdot 1.05 \cdot 1.05$. 

After three years, the money in the bank would be $110.25 + 110.25 \cdot 1.05 = 115.76$. You make $5$ in interest the first year, $5.25$ in the second year, and $5.51$ in the third year. The important point is that you are making interest on interest. While this doesn't seem significant, we'll see otherwise shortly.
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You make $5 in interest the first year, $5.25 in the second year, and $5.51 in the third year. The important point is that you are making interest on interest. While this doesn’t seem significant, we’ll see otherwise shortly.
In general, putting $100 into the bank and leaving it alone, after $n$ years the money you’d have in the bank is

$$100 \cdot (1.05)^n$$
Entering on a Calculator

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To enter $100 \cdot (1.05)^n$ on a scientific calculator, enter

$$100 \times (1.05)^n =$$

or

$$100 \times (1.05)^n =$$

depending if you calculator uses $\times$ or $\times$ for multiplication and if it has a $\wedge$ button or a $y^x$ button.
We saw that as time goes on you make more interest than in the beginning years. However, over three years the increase in interest seems pretty trivial.
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The increase in interest shows up the longer you let the money grow. The next chart gives a pictorial representation of the money in your account if you start with $100 and leave the account untouched.
Compound Interest, 5% Per Year

Year 1: $100.00
Year 2: $105.00
Year 3: $110.25
Year 4: $115.76
Year 5: $121.55
Year 6: $127.56
Year 7: $133.90
Year 8: $140.52
Year 9: $147.49
Year 10: $154.87
Year 11: $162.67
Year 12: $170.84
Year 13: $179.42
Year 14: $188.42
Year 15: $197.80
Year 16: $207.60

Total after 16 years: $207.60

Interest Rates 23 April 2012
If you put $100 in the bank at 5%, compounded yearly, how much will you have after 200 years? Put in the number into your clicker.

To make things simpler, just put in the number of dollars, ignore cents.

If you have a computer, check out web2.0calc.com for doing calculator computations.
After 200 years you’d have

$$100 \cdot 1.05^{200} = 1,729,258!$$
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If we get 6% per year, we’d end up with

\[ 100 \cdot 1.06^{200} = 11,512,590. \]
Compound Interest, 5% per year

$0

$2,000,000

$1,800,000

$1,600,000

$1,400,000

$1,200,000

$1,000,000

$800,000

$600,000

$400,000

$200,000

$0

10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200
What is happening is that, each 10 years, your money increases by about 63%. The height of a bar is then 63% larger than the previous one. This may not seem like a big deal at first, but once you have a lot of money, it is really significant.
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For example, while you make $63 in the first 10 years, if you had $1M in the bank instead of $100, you’d make about $630,000 in interest in the first 10 years.
A General Formula

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Let’s call the length of time the bank compounds as the period. If you put money in the bank at a given interest rate per year, converted to a decimal, then the formula for how much money you have after so many periods is

$$\text{future amount} = \text{principal} \cdot (1 + \frac{\text{interest rate}}{\# \text{ periods per year}})^{\text{number of periods}}$$
In symbols, the formula is

\[ F = P \cdot \left(1 + \frac{r}{n}\right)^m \]

where

- \( F \) = future amount
- \( P \) = principal (initial investment)
- \( r \) = interest rate per year (converted to decimal)
- \( n \) = number of periods per year
- \( m \) = number of periods gathering interest

The fraction \( r/n \) is the interest rate per period.
Spreadsheets allow one to do multiple calculations more easily. There are also computer programs and webpages which are set up to do interest rate calculations.
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At the website


you can do all the various financial calculations.
1 If you invest $1,000 at 6% per year, compounded daily, how much do you have in 5 years?

Ans. Your period is 1 day, so your interest rate per day is \(0.06/365\). In 5 years there are \(365 \times 5 = 1,825\) days. So, the amount you will have is 

\[ \$1000 \times \left(1 + \frac{0.06}{365}\right)^{1825} = \$1,349.83 \]

2 What if you get 1% per year, compounded daily?

Ans. You will have 

\[ \$1000 \times \left(1 + \frac{0.01}{365}\right)^{1825} = \$1,051.27 \]
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They come up because different organizations can compound differently. The APR or APY represents what interest rate, compounded yearly, that would be equivalent to a given rate compounded in any time period.
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Using APR gives a way to compare interest rates that are compounded differently.
For example, suppose we have a 6% interest rate compounded daily. What is the APR? If we ask how much money we’ll have if we invest $100 for a year, we’d have

\[
100 \cdot \left(1 + \frac{0.06}{365}\right)^{365} = 106.18
\]

If we compounded just once a year, we’d need an interest rate of 6.18% to end up with the same amount. The APR is then 6.18%.
More generally, if we invest $P$ with an annual interest rate $r$ compounded $n$ times per year, if $APR$ is the annual percentage rate, then looking at the amount we’d have after 1 year, we’d would have

$$P \cdot \left(1 + \frac{r}{n}\right)^n = P \cdot (1 + APR)$$
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If we cancel the $P$ from both sides and subtract 1, we’ll get

$$APR = \left(1 + \frac{r}{n}\right)^n - 1$$
Doubling Time

In an earlier example we saw that, with 5% compounded yearly, each 10 years our money increases by 63%. More generally, the amount of time it takes for money to increase a certain percentage does not depend on how much you invest.
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In particular, we can ask how long it takes for money to double (increase by 100%).

For example, if we get 5% compounded per year, then how many years \( n \) does it take to double $100? We’d have to solve

\[
100 \cdot (1.05)^n = 200
\]

for \( n \). We can do this with trial and error, using the Interest Calculator spreadsheet, and we’ll get a little over 14 years.
More generally, if \( r \) is our interest rate per year, \( n \) is how many periods per year, and \( y \) is the number of years it takes for money to double at that rate, we’d ask for the value of \( y \) for which

\[
P \cdot \left(1 + \frac{r}{n}\right)^{ny} = 2P
\]

Canceling \( P \) gives

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Canceling $P$ gives

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For those who have seen logarithms, taking logs of both sides and simplifying gives

$$y = \frac{\log(2)}{n \cdot \log \left(1 + \frac{r}{n}\right)}$$
Here is a table that gives doubling times for a few interest rates.

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For example, if your rate was 15%, and you started with $100, in 5 years you’d have about $200, in 10 years about $400, in 15 years about $800, and so on.
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Doubling Rule of Thumb

The Ninth Wonder of the world!

\[
\frac{72}{\text{interest rate \%}} = \text{# of years for your money to Double!}
\]
Most people don’t use the formula for doubling time. Instead, here is a simple rule of thumb which gives a pretty good approximation.
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To find the doubling time, divide the yearly interest rate percentage into 72. The result is about how many years it takes for money to double.
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To find the doubling time, divide the yearly interest rate percentage into 72. The result is about how many years it takes for money to double.

For example, if your interest rate is 6% per year, then your money will double in about

\[
\frac{72}{6} = 12 \text{ years}
\]
Next Time

We’ll look at inflation and start to talk about loans.
If you put money in the bank at an annual interest rate of 2%, how long will it take for your money to double?

A  1 year

B  10 years

C  36 years

D  Never