Poker: Probabilities of the Various Hands

19 February 2014
There are 4 suits and 13 values. The suits are
- Spades ♠
- Hearts ♥
- Diamonds ♦
- Clubs ♣

There are 13 values: 2 through 10, Jack, Queen, King, and Ace.

2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
Poker Hands

A poker hand is made up of 5 cards. The different poker hands are, from best to worst, are:

<table>
<thead>
<tr>
<th>Hand</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Royal Flush</td>
<td>(A\spadesuit, K\spadesuit, Q\spadesuit, J\spadesuit, 10\spadesuit)</td>
</tr>
<tr>
<td>Straight Flush</td>
<td>(J\heartsuit, 10\heartsuit, 9\heartsuit, 8\heartsuit, 7\heartsuit)</td>
</tr>
<tr>
<td>4 of a kind</td>
<td>(8\spadesuit, 8\heartsuit, 8\diamondsuit, 8\clubsuit, 7\heartsuit)</td>
</tr>
<tr>
<td>Full House</td>
<td>(4\heartsuit, 4\diamondsuit, 4\clubsuit, J\spadesuit, J\clubsuit)</td>
</tr>
<tr>
<td>Flush</td>
<td>(K\spadesuit, 10\spadesuit, 8\spadesuit, 7\spadesuit, 3\spadesuit)</td>
</tr>
<tr>
<td>Straight</td>
<td>(Q\heartsuit, J\diamondsuit, 10\spadesuit, 9\clubsuit, 8\heartsuit)</td>
</tr>
<tr>
<td>3 of a kind</td>
<td>(8\heartsuit, 8\diamondsuit, 8\clubsuit, J\heartsuit, 6\spadesuit)</td>
</tr>
<tr>
<td>2 pair</td>
<td>(K\spadesuit, K\heartsuit, 9\spadesuit, 9\diamondsuit, A\spadesuit)</td>
</tr>
<tr>
<td>1 pair</td>
<td>(7\heartsuit, 7\diamondsuit, 10\clubsuit, J\heartsuit, 4\spadesuit)</td>
</tr>
</tbody>
</table>
There are nearly 2.6 million poker hands. If $nC_r$ represents the number of ways to choose $r$ things out of $n$, then $52C_5$ is the number of poker hands.

The formula for $nC_r$ is

$$nC_r = \frac{n!}{r! \cdot (n - r)!}$$

The explanation point is spoken factorial; that is, $5!$ is spoken as 5 factorial. The numbers $nC_r$ are called combinations or binomial coefficients.
The factorial $n!$ means to multiply all the whole numbers between 1 and $n$. For example,

\begin{align*}
1! &= 1 \\
2! &= 2 \cdot 1 = 2 \\
3! &= 3 \cdot 2 \cdot 1 = 6 \\
4! &= 4 \cdot 3 \cdot 2 \cdot 1 = 24 \\
5! &= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120
\end{align*}

Scientific calculators have a factorial button. More basic calculators do not. There are websites for calculating $nC_r$. 
Blaise Pascal, a mathematician and physicist, lived in the 17th century. We heard his name last week with regard to the origins of probability theory. Perhaps he is best known today for the so-called Pascal’s Triangle:
The picture above shows the first 9 rows of Pascal’s Triangle. The full triangle goes on forever.

The significance of the triangle is it contains the various binomial coefficients. The numbering convention for using it is that the rows are numbered $0, 1, 2, \ldots$, and the entries, from left to right, in a row are numbered $0, 1, 2, \ldots$. If you want $nC_r$, you go to the $n$-th row and find the $r$-th entry of that row; this gives you $nC_r$. 
One can generate as many rows of the triangle as one wishes. The way to build a new row is to start and end with the number 1. Then, for any other number, add the two numbers immediately above the one you want.
Last time we computed how many ways are there to choose $r$ things out of 4 for all possible $r$ (except for 0). The numbers we got were

<table>
<thead>
<tr>
<th>$r$</th>
<th>$4 \binom{4}{r}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

These are the numbers in row 4 of Pascal’s triangle.
The reason these numbers are called binomial coefficients is because they arise as coefficients in expanding \((x + y)^n\) into binomials. For example,

\[(x + y)^2 = x^2 + 2xy + y^2\]

and

\[(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\]

and

\[(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\]

The entries in row \(n\) of Pascal’s triangle gives the coefficients that show up in expanding \((x + y)^n\).
We’ll now look at what is the probability of being dealt a given poker hand when dealt 5 cards at random. Each hand of 5 cards is just as likely as any other to be dealt. Since there are 2,598,960 total hands, to compute the probability we need to compute how many of these represent the given type of hand.

We can then use the formula

\[
\text{probability} = \frac{\text{number of ways the outcome can occur}}{\text{total number of outcomes}}.
\]
What is the probability of getting a royal flush when dealt 5 cards? Let’s first ask: how many different royal flushes there are? Enter the number you think with your clicker.

There are 4 ways to get a royal flush, since the only choice is which suit you get. They are

- A♠ K♠ Q♠ J♠ 10♠,
- A♥ K♥ Q♥ J♥ 10♥,
- A♦ K♦ Q♦ J♦ 10♦,
- A♣ K♣ Q♣ J♣ 10♣

So, the probability is \( \frac{4}{2598960} \), or about 1 out of 600,000 deals.
Probability of a Straight Flush

There are few enough straight flushes that we can list them all. This is a bit tedious, but not too bad. Here they are:

<table>
<thead>
<tr>
<th>A♥2♥3♥4♥5♥</th>
<th>A♥2♥3♥4♥5♥</th>
<th>A♥2♥3♥4♥5♥</th>
<th>A♥2♥3♥4♥5♥</th>
</tr>
</thead>
<tbody>
<tr>
<td>2♥3♥4♥5♥6♥</td>
<td>2♥3♥4♥5♥6♥</td>
<td>2♥3♥4♥5♥6♥</td>
<td>2♥3♥4♥5♥6♥</td>
</tr>
<tr>
<td>3♥4♥5♥6♥7♥</td>
<td>3♥4♥5♥6♥7♥</td>
<td>3♥4♥5♥6♥7♥</td>
<td>3♥4♥5♥6♥7♥</td>
</tr>
<tr>
<td>4♥5♥6♥7♥8♥</td>
<td>4♥5♥6♥7♥8♥</td>
<td>4♥5♥6♥7♥8♥</td>
<td>4♥5♥6♥7♥8♥</td>
</tr>
<tr>
<td>5♥6♥7♥8♥9♥</td>
<td>5♥6♥7♥8♥9♥</td>
<td>5♥6♥7♥8♥9♥</td>
<td>5♥6♥7♥8♥9♥</td>
</tr>
<tr>
<td>6♥7♥8♥9♥10♥</td>
<td>6♥7♥8♥9♥10♥</td>
<td>6♥7♥8♥9♥10♥</td>
<td>6♥7♥8♥9♥10♥</td>
</tr>
<tr>
<td>7♥8♥9♥10♥J♥</td>
<td>7♥8♥9♥10♥J♥</td>
<td>7♥8♥9♥10♥J♥</td>
<td>7♥8♥9♥10♥J♥</td>
</tr>
<tr>
<td>8♥9♥10♥J♥Q♥</td>
<td>8♥9♥10♥J♥Q♥</td>
<td>8♥9♥10♥J♥Q♥</td>
<td>8♥9♥10♥J♥Q♥</td>
</tr>
<tr>
<td>9♥10♥J♥Q♥K♣</td>
<td>9♥10♥J♥Q♥K♣</td>
<td>9♥10♥J♥Q♥K♣</td>
<td>9♥10♥J♥Q♥K♣</td>
</tr>
</tbody>
</table>

There are 36 straight flushes. Note that we are not counting the royal flushes.
Some General Ideas About Counting

Writing down all the straight flushes wasn’t too bad, but it isn’t a great general method.

In order to compute the probability of other hands, one approach is to decide what things you need to choose in order to write down a hand, and then determine how many ways each of the choices can occur.

For example, to count royal flushes, you only have to choose a suit because we must then take $10, J, Q, K, A$ of that suit.
If one event does not affect the outcome of another, they are called independent. To count the number of ways a pair of independent events can occur, multiply the number of ways each way can occur.

This is basically the same mathematical idea as multiplying probabilities in a probability tree we used last week.

Rolling two dice is an example of two independent events: what you get on one die does not affect what can happen on the other. Since there are 6 outcomes for rolling one die, there are \(6 \cdot 6 = 36\) outcomes for rolling two dice.
How can we use the idea above to count the number of straight flushes?

To choose a given straight flush, you must choose a suit, and a starting (or ending) value for the 5 in a row. There are 4 choices for the suit.

**Q** How many ways are there to choose the starting value of a straight flush (which is not a royal flush)? Enter the number with your clicker.

**A** There are 9 values. It would appear that there are 10 possible starting values (A through 10). However, if we want a straight flush which is not a royal flush, we cannot start at 10, so there are 9 choices.
Choosing the suit and the starting value are independent events. So, to count the number of straight flushes, we need to multiply the number of choices of suit and the number of choices of starting value.

Therefore, there are $4 \cdot 9 = 36$ total straight flushes.

The probability of a straight flush (which is not a royal flush) is then $36/2,598,960$, which is about 1 out of 72,000.
What is the probability of a 4 of a kind? An example is

\[8\spadesuit, 8\heartsuit, 8\diamondsuit, 8\clubsuit, J\heartsuit\]

To get a 4 of a kind, you must choose the value of the 4 of a kind, and choose the remaining card.

There are 13 choices for the value of the 4 of a kind.

Q How many choices are there for the remaining card?

A 48. The 5th card can be any of the remaining 48 cards.
Because we can view writing down a 4 of a kind as choosing the value of the 4 of a kind and picking the last card, which are independent events, we need to multiply the number of choices together.

The number of four of a kinds is then $13 \cdot 48 = 624$. The probability of a 4 of a kind is then $624/2598960$, or about 1 out of 4200.
A full house consists of a 3 of a kind and a 2 of a kind. What is the probability of getting a full house? An example is

\[4\heartsuit, 4\diamondsuit, 4\clubsuit, J\spadesuit, J\heartsuit\]

To have a full house you must choose the value of a 3 of a kind and the value of a 2 of a kind. You must also choose which 3 cards make up the 3 of a kind and which 2 make up the 2 of a kind. This is one of the more complicated counts.
Q How many ways are there to choose the value of the 3 of a kind?

A There are \( 13 \binom{1}{} = 13 \) ways to choose the value of the 3 of a kind.

Q How many ways are there to choose the value of the 2 of a kind?

A There are \( 12 \binom{1}{} = 12 \) ways to choose the value of the pair. You can choose any value other than the value of the 3 of a kind.
For sake of argument, say we choose Jacks for the 3 of a kind and 7 for the 2 of a kind. How many ways are there to choose the 3 jacks?

There are $4 \binom{3}{3} = 4$ ways to choose the 3 Jacks. They are:

- $\spadesuit J \spadesuit J \diamondsuit$
- $\spadesuit J \spadesuit J \heartsuit$
- $\diamondsuit J \heartsuit J \spadesuit$
- $\diamondsuit J \heartsuit J \heartsuit$

How many ways are there to choose the two 7s?

There are $4 \binom{2}{2} = 6$ ways to choose the two 7s. They are:

- $\spadesuit 7 \heartsuit 7$
- $\spadesuit 7 \diamondsuit 7$
- $\spadesuit 7 \spadesuit 7$
- $\heartsuit 7 \heartsuit 7$
- $\heartsuit 7 \diamondsuit 7$
- $\diamondsuit 7 \diamondsuit 7$
To find the number of full houses, we then have to multiply the numbers of our various choices (value of the 3 of a kind, value of the pair, the three cards for the 3 of a kind, the two cards for the pair).

So, the number of full houses is

$$13 \cdot 12 \cdot 4 \cdot 6 = 3744$$

The probability of a full house is then $3744/2598960$, which is about 1 out of 4000 hands.
In order to have a flush, which is 5 cards of the same suit, we need to choose the suit, and then pick the five cards.

There are 4 choices for the suit.

Q Suppose we decide to get a flush with hearts. How many ways are there to pick 5 hearts?

A We need to pick 5 of the 13 hearts. The number of ways is $\binom{13}{5}$. This is equal to 1287.

The number of flushes is then $4 \cdot 1287 = 5148$. However, this includes royal and straight flushes. Subtracting the 40 of them gives 5108 flushes. This means the probability of being dealt a flush is $\frac{5108}{2598960}$, or about 1 out of every 500 hands.
We could continue with the remaining types of hands. But, instead we summarize the results:

<table>
<thead>
<tr>
<th>Type of Hand</th>
<th>Number of that Type</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Royal Flush</td>
<td>4</td>
<td>0.00015%</td>
</tr>
<tr>
<td>Straight Flush</td>
<td>36</td>
<td>0.001%</td>
</tr>
<tr>
<td>4 of a Kind</td>
<td>624</td>
<td>0.02%</td>
</tr>
<tr>
<td>Full House</td>
<td>3,744</td>
<td>0.15%</td>
</tr>
<tr>
<td>Flush</td>
<td>5,108</td>
<td>0.2%</td>
</tr>
<tr>
<td>Straight</td>
<td>10,200</td>
<td>0.4%</td>
</tr>
<tr>
<td>3 of a Kind</td>
<td>54,912</td>
<td>2.1%</td>
</tr>
<tr>
<td>2 Pair</td>
<td>123,552</td>
<td>4.8%</td>
</tr>
<tr>
<td>1 Pair</td>
<td>1,098,240</td>
<td>42.3%</td>
</tr>
<tr>
<td>Nothing</td>
<td>1,302,540</td>
<td>50.1%</td>
</tr>
</tbody>
</table>
We will discuss some probabilities involved in playing poker. More specifically, we’ll look at the game **Texas Hold’em**, which is perhaps the most popular poker game. We will introduce the notion of **expected value**, which helps to determine good strategy. We’ll use this idea next week when we discuss lotteries.