In 2014 every member of the house is up for election and about a third of the senate seats will be up for grabs.

Most people do not realize that there is mathematics involved in voting, besides counting ballots. Most people also do not realize that there are many different voting systems, used both in the U.S. and in other countries. We will discuss four voting systems and problems with each.
Voting theory has been considered for centuries. However, it appears to have become a concern of theoretical study in the late 1700’s. Jean-Charles de Borda and the Marquis de Condorcet are often credited with founding voting theory. Borda proposed a voting system, which we will study. Condorcet discovered some problems and paradoxes of voting systems, which we will also study.

We’ll first check out a Simpson’s video about voting in the 2008 presidential election.

2008 Voting
Plurality Voting

The method we use in the U.S. to decide most elections is called Plurality Voting. In this method, a voter chooses one candidate, and the candidate with the most votes wins. It is nearly the most simple method of deciding elections.

Plurality voting satisfies the following three properties:

1. All voters are treated the same. If two voters were to exchange ballots before turning them in, this would not affect the outcome.

2. All candidates are treated the same. If every voter were to reverse their preference for two candidates, the outcome would be reversed.

3. If a single voter were to change their ballot from being for the loser to the winner, and everybody else were to keep their vote the same, the outcome would not change.
Kenneth May, an American mathematician, proved in 1952 that plurality voting is the only voting system which satisfies all three of the properties on the previous page.

One place where plurality voting isn’t used is in the U.S. presidential election. The Electoral College is used.
Each State shall appoint, in such Manner as the Legislature thereof may direct, a Number of Electors, equal to the whole Number of Senators and Representatives to which the State may be entitled in the Congress.

The Electors shall meet in their respective States, and vote by Ballot for two Persons, of whom one at least shall not be an Inhabitant of the same State with themselves. The Person having the greatest Number of Votes shall be the President, if such Number be a Majority of the whole Number of Electors appointed.
The Electoral College 2012

[Map showing electoral college votes for 2012, with states color-coded to indicate gains, losses, or no change.]
There are 100 senators, 2 per state. There are 435 members of the House of Representatives. Then there are 538 electoral college votes; Washington D.C. gets 3 electoral college votes even though it has no congressional representation. If a presidential candidates receives at least 270 electoral college votes, they’ll have more than half, so will win the election.

The balance of electoral college votes depends on the population of the country. Each ten years a census is conducted, and the number of electoral college votes a state has can change. The map on the previous page shows the change from 2000 to 2010.

While plurality voting in presidential elections isn’t used for the population as a whole, it is used among electoral college votes.
Elections with Two Candidates

If an election has just two candidates, there isn’t much of an issue. It turns out that every reasonable voting system will give the same outcome, so using plurality voting is the most sensible thing to do.

As we will see, complications with this method occur when there are more than two viable candidates. We will look at several examples of elections, how they came out, and how other voting systems would have affected the outcome.
Elections with Three or More Candidates

There are many situations where an election has more than two candidates. Democratic and Republican state primaries are just two such examples. Even the U.S. presidential election has had more than two candidates who received a fair portion of the votes. This happened in 1992 and in 2000. In both cases the presence of a third party candidate possibly affected the outcome of the election.
Let’s Have a Vote

Use your clicker to vote for one of the following Marvel Comic characters from the movie The Avengers.

A  Iron Man
B  Captain America
C  The Hulk
D  Thor
E  Black Widow
Some Examples of Plurality Voting

Perhaps the election which most clearly shows issues with plurality voting is the 1998 Minnesota gubernatorial election.

In that election, Jesse Ventura defeated Hubert Humphrey III and Norm Coleman.
Jesse “The Body” Ventura

Jesse Ventura was a former navy seal, pro wrestler, and actor before he turned to politics.
Ventura acted in the 1987 movie Predator. The star of the movie was Arnold Schwarzenegger, who went on to become Governor of California in 2003.

It remains to be seen if any other actor from the movie is elected Governor of some state. At least one other actor from the movie has run for governor.
In the 1998 Minnesota governors race, Jessie Ventura defeated Hubert Humphrey III and Norm Coleman. Ventura received 37% of the vote to Coleman’s 35% and Humphrey’s 28%. Most of those who did not vote for Ventura, when polled after the election, indicated their strong disapproval of the election of Ventura. Thus, nearly 2/3 of the voters were unhappy with the outcome.

A variant of plurality voting is to have runoff elections. In the main election, if no candidate receives over 50% of the votes, the top two candidates compete in a runoff election, and the candidate who gets the most votes in the runoff is elected. Many countries use runoffs to decide presidential elections.
Runoff elections are used in many places in the U.S., including statewide elections. Lots of cities also use runoff elections.

If Minnesota used a runoff election, then Coleman and Ventura would have competed. Coleman would have almost certainly been elected in this case.
This race featuring Al D’Amato, Elizabeth Holtzman, and Jacob Javits. D’Amato received 45%, Holtzman 44% and Javits 11%, so D’Amato was elected.

Based on exit polls, if voters were to rank the three candidates, then an estimate of the outcome is given in the following table.

<table>
<thead>
<tr>
<th>Rank</th>
<th>22%</th>
<th>23%</th>
<th>15%</th>
<th>29%</th>
<th>7%</th>
<th>4%</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>D</td>
<td>D</td>
<td>H</td>
<td>H</td>
<td>J</td>
<td>J</td>
</tr>
<tr>
<td>Second</td>
<td>H</td>
<td>J</td>
<td>D</td>
<td>J</td>
<td>H</td>
<td>D</td>
</tr>
<tr>
<td>Third</td>
<td>J</td>
<td>H</td>
<td>J</td>
<td>D</td>
<td>D</td>
<td>H</td>
</tr>
</tbody>
</table>
According to this table, 49% preferred D’Amato to Holtzman, while 51% preferred Holtzman to D’Amato. Thus, D’Amato was elected even though a majority preferred another candidate.

Holtzman is called a Condorcet winner, since she was the preferred candidate based on head to head competition with each candidate.

A voting system satisfies the Condorcet winner criterion if the Condorcet winner, if there is one, always wins the election.

The two examples we’ve considered shows that plurality voting therefore does not satisfy the Condorcet winner criterion.
Suppose we have the following election between three candidates:

<table>
<thead>
<tr>
<th>Rank</th>
<th>Votes</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>A</td>
</tr>
<tr>
<td>Second</td>
<td>B</td>
</tr>
<tr>
<td>Third</td>
<td>C</td>
</tr>
</tbody>
</table>

Then there is no Condorcet winner since A beats B head to head, B beats C head to head, while C beats A head to head. So, no matter who is elected, 2/3 of the public prefers somebody else to the winner. This is called the Condorcet voting paradox, meaning that, collectively, A can be preferred to B, who is preferred to C, but C is preferred to A. Symbolically, $A > B > C > A$. 
Third party candidates affected the U.S. presidential election in significant ways in both 1992 and 2000. In 1992, the main candidates were George Bush, Bill Clinton, and Ross Perot. Clinton received 43% of the popular vote, Bush received 37%, and Perot 19%. Since Perot’s support came mostly from Republicans, many believe that his presence in the race was the reason Clinton won.

In 2000, the main candidates were George W. Bush, Al Gore, and Ralph Nader. Bush won 47.9% of the popular vote, Gore won 48.4%, and Nader 2.7%. The election came down to Florida.

There Bush received 2,912,790 votes, Gore received 2,912,253 votes, and Nader 97,488. Out of 6 million votes in Florida, Bush and Gore were separated by only 637 votes. Since Nader’s support came mostly from Democrats, many believe that his presence cost Gore from winning Florida, which then cost him the election.
We will discuss three other voting systems, each of which is used in various places. They all have voters rank all the candidates rather than vote for just a single candidate. These systems are the Borda count, Sequential Pairwise Voting, and the Hare (or instant runoff) system.
The Borda Count

The **Borda count**, created by Jean-Charles de Borda, is commonly used in sports, along with being used in a few countries. For example, it is used to pick the Heisman Trophy winner, the most valuable player in professional baseball, and make various NCAA rankings.

In this system voters rank order the candidates. If there are $n$ candidates, each first place vote is worth $n - 1$ points, each second place vote is worth $n - 2$ points, and so on, down to 0 points for last place votes.

The person who received the most points wins the election.
For example, suppose that there are three candidates, which we will list as A, B, and C. Suppose that 60% lists A first, B second, and C third, and the remaining 40% lists C first, B second, and A third.

To make the arithmetic easier, let’s assume there are 10 ballots.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Number of Voters</th>
</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>A</td>
</tr>
<tr>
<td>Second</td>
<td>B</td>
</tr>
<tr>
<td>Third</td>
<td>C</td>
</tr>
</tbody>
</table>

With 3 candidates, a first place vote gets 2 points, a second place vote gets 1 point, and a third place vote gets 0 points.
Q  How many points does A receive?

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A  Candidate A receives 2 points for each of his 6 first place votes, and 0 for each of the 4 third place votes. His total is then 12 points.
Q How many points does B receive?

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>First</td>
<td>A</td>
</tr>
<tr>
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<td>B</td>
</tr>
<tr>
<td>Third</td>
<td>C</td>
</tr>
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Candidate B receives 1 point each for all of her 10 second place votes. She then has a total of 10 points.
Next time we’ll continue this example, and we’ll also look at two more voting systems, Sequential Pairwise Voting and the Hare System.