Theorem 1. Let \( f : X \to Y \) be an onto function. Then there is a function \( g : Y \to X \) with \( f \circ g = \text{id}_Y \).

Proof. Because \( f \) is onto, for each \( y \in Y \) there are elements in \( X \) mapping to \( y \). In particular, \( f^{-1}(y) := \{ x \in X : f(x) = y \} \) is a nonempty set for each \( y \in Y \). For each \( y \in Y \) choose an \( x_y \in X \) with \( f(x_y) = y \). Define a function \( g : Y \to X \) by \( g(y) = x_y \). Then \((f \circ g)(y) = f(g(y)) = f(x_y) = y\), by the choice of \( x_y \). Thus, \( f \circ g = \text{id}_Y \). \( \square \)

While this proof seems simple, and can often times be found in Math 279, it hides an important idea. The argument requires us to be able to “choose” an element in the set \( f^{-1}(y) \) for each \( y \in Y \). While you may think this is an obvious fact, from the most common set of axioms used in set theory, the Zermelo-Fraenkel axioms (which can be found searching for Zermelo-Fraenkel set theory in Wikipedia), the axiom of choice is independent, meaning that it cannot be proved or disproved from the other axioms. Many mathematicians do not like using the axiom of choice. One problem people have with it is that it has some unusual consequences. For example, Banach and Tarski proved in 1924 that, by using the axiom of choice, any solid sphere can be decomposed into finitely many subsets which can themselves be reassembled to form two solid spheres, each of the same size as the original (see http://publish.uwo.ca/ jbell/CHOICE.pdf).

Since the time of Banach and Tarski much has been done on how the axiom of choice relates to other mathematical results. It is known that the axiom of choice is equivalent to Zorn’s Lemma, and that it is also equivalent to: For every field \( F \) and every \( F \)-vector space \( V \), then \( V \) has a basis.