Normed Vector Spaces and Double Duals

Mathematics 481/525

In this note we look at a number of infinite-dimensional $\mathbb{R}$-vector spaces that arise in
analysis, and we consider their dual and double dual spaces. As an application, we give an
example of an infinite-dimensional vector space $V$ for which the natural map $\eta : V \to V^{**}$ is
not an isomorphism. In analysis, duals and double duals of vector spaces are often defined
differently than in algebra, by considering continuity. We will be more specific shortly.

Let $V$ be an $\mathbb{R}$-vector space. We say that $V$ is a normed vector space if there is a function
$\| \cdot \| : V \to \mathbb{R}$ which satisfies

- $\|v\| \geq 0$ for all $v \in V$, and $\|v\| = 0$ if and only if $v = 0$.
- $\|\alpha v\| = |\alpha| \cdot \|v\|$ for all $\alpha \in \mathbb{R}$ and $v \in V$.
- $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in V$.

The function $\| \cdot \|$ is called a norm on $V$. If $V$ is a normed vector space, then the function
$\| \cdot \|$ allows us to define a metric on $V$ by $d(v, w) = \|v - w\|$, just as we do for $\mathbb{R}$ or $\mathbb{R}^n$. We
can then talk about functions $f : V \to \mathbb{R}$ being continuous at $v_0 \in V$: if for any $\varepsilon > 0$ there
exists $\delta > 0$ such that $\|v - v_0\| < \delta$ implies $\|f(v) - f(v_0)\| < \varepsilon$, then $f$ is continuous at $v_0$.

**Example 1.** If $V = \mathbb{R}^n$, then $V$ is a normed vector space under the usual Euclidean metric
$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Recall that the set $S$ of all real valued sequences is an $\mathbb{R}$-vector space under pointwise
addition and scalar multiplication. The next example gives a collection of subspaces of this
sequence space.

**Example 2.** For $x = \{x_n\} \in S$, define, for $p \geq 1$,

$$
\|x\|_p = \left( \sum_{n=1}^{\infty} |x_n|^p \right)^{1/p}
$$

and

$$
\|x\|_\infty = \sup_n \{|x_n|\}.
$$
If \( p = 1 \), then \( \|x\|_1 = \sum_{n=1}^{\infty} |x_n| \). By the convention that \( \infty + \infty = \infty \) and \( r < \infty \) for each \( r \in \mathbb{R} \), we see that each of these three functions satisfy all properties of a norm except possibly for \( \|x\| < \infty \). Set

\[
  l^p = \left\{ x \in S : \|x\|_p < \infty \right\},
  l^\infty = \left\{ x \in S : \|x\|_\infty < \infty \right\}.
\]

Then each of these are normed vector spaces. Furthermore, let

\[
c_0 = \left\{ \{x_n\} \in l^\infty : \lim_{n \to \infty} x_n = 0 \right\}.
\]

Then \( c_0 \) is a subspace of \( l^\infty \), and so \( c_0 \) is a normed vector space with respect to the norm \( \| \cdot \|_\infty \). A short argument shows that if \( p \leq p' \), then \( l^p \subseteq l^{p'} \). We then have the containments \( l^1 \subseteq l^p \subseteq c_0 \subseteq l^\infty \) for each \( p \geq 1 \).

**Example 3.** If \((X, \mu)\) is a measure space, \( p \geq 1 \), and \( V \) is the vector space of all measurable functions \( X \to \mathbb{R} \), then \( V \) is a normed vector space under the norm

\[
  \|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p}.
\]

To make \( \| \cdot \|_p \) a norm, we must identify functions that agree almost everywhere, since \( \|f\|_p = 0 \) if and only if \( f = 0 \) a.e. A proof of the triangle inequality can be found in [1, Thm 3.5].

Let \( V \) be a normed vector space. If \( T : V \to \mathbb{R} \) is a linear functional, define

\[
  \|T\| = \sup \{ |T(x)| : x \in V, \|x\| = 1 \}.
\]

We say \( T \) is **bounded** if \( \|T\| < \infty \). We note that if \( T \) is linear and \( x \in V \) is nonzero, then we may write \( x = \alpha y \) with \( \alpha = \|x\| \) and \( y = x/\|x\| \). Then \( T(x) = \alpha T(y) = \|x\| T(y) \). Consequently, \( |T(x)| / \|x\| = |T(y)| \). Thus,

\[
  \|T\| = \sup \left\{ \frac{|T(x)|}{\|x\|} : x \in V, x \neq 0 \right\}.
\]

As a consequence, \( |T(x)| \leq \|T\| \cdot \|x\| \) for all \( x \in V \). More generally, this calculation shows that if \( x = \alpha y \) for any nonzero \( \alpha \in \mathbb{R} \), then \( |T(x)| / \|x\| = |T(y)| / \|y\| \).

**Lemma 4.** Let \( T : V \to \mathbb{R} \) be a linear functional. Then the following statements are equivalent.

1. \( T \) is bounded.
2. \( T \) is uniformly continuous.
3. \( T \) is continuous.
(4) \( T \) is continuous at some \( v_0 \in V \).

Proof. (1) implies (2): Suppose that \( T \) is bounded. Let \( \varepsilon > 0 \) and take \( v, w \in V \). Then
\[
|T(w) - T(v)| = |T(w-v)| \leq \|T\| \cdot \|w-v\|.
\]
Thus, if we define \( \delta = \varepsilon / \|T\| \), this calculation shows that \( \|w-v\| < \delta \) implies \( |T(w) - T(v)| < \varepsilon \). Therefore, \( T \) is uniformly continuous.

(2) implies (3) and (3) implies 4) are both trivial.

(4) implies (1): Suppose \( T \) is continuous at \( v_0 \). Then for \( \varepsilon = 1 \), there is a \( \delta > 0 \) such that if \( \|v-v_0\| < \delta \), then \( \|T(v) - T(v_0)\| < 1 \). By setting \( x = v - v_0 \) and noting that \( T(v) - T(v_0) = T(x) \), we see that if \( \|x\| < \delta \), then \( \|T(x)\| < 1 \). Let \( v \in V \) be nonzero and let \( x = \frac{\delta}{2\|v\|} v \). Then \( \|x\| = \delta/2 \), so \( |T(x)| < 1 \). Since \( |T(x)| / \|x\| = |T(v)| / \|v\| \), we see that \( |T(v)| / \|v\| \leq 2/\delta \). This implies that \( \|T\| \leq 2/\delta \), so \( T \) is bounded.

Let
\[ \text{hom}_b(V, \mathbb{R}) = \{ T \in \text{hom}(V, \mathbb{R}) : \|T\| < \infty \} . \]
The set \( \text{hom}_b(V, \mathbb{R}) \) consists of all bounded linear functionals on \( V \). By the lemma, this set is the same as the set of all continuous linear functionals on \( V \). Since the sum and difference of continuous maps is continuous, and any scalar multiple of a continuous map is continuous, we see that \( \text{hom}_b(V, \mathbb{R}) \) is a subspace of \( \text{hom}(V, \mathbb{R}) \). We consider \( \text{hom}_b(V, \mathbb{R}) \) to be the analytic dual space of \( V \).

Lemma 5. If \( V \) is a normed vector space, then \( \text{hom}_b(V, \mathbb{R}) \) is a normed vector space under the definition of the norm \( \|T\| \) given in Equation (1) above.

Proof. It is clear that \( \|T\| \geq 0 \) for any \( T \in \text{hom}_b(V, \mathbb{R}) \), and that if \( \|T\| = 0 \), then \( T(x) = 0 \) for all \( x \in V \), so \( T = 0 \). Next, let \( \alpha \in \mathbb{R} \). Then
\[
\|\alpha T\| = \sup \{ |\alpha T(x)| : \|x\| = 1 \} = \sup \{ |\alpha| \cdot |T(x)| : \|x\| = 1 \}
= |\alpha| \cdot \sup \{ |T(x)| : \|x\| = 1 \} = |\alpha| \cdot \|T\| .
\]
Finally, if \( S, T \in \text{hom}_b(V, \mathbb{R}) \), then
\[
\|S + T\| = \sup \{ |S(x) + T(x)| : \|x\| = 1 \} \leq \sup \{ |S(x)| + |T(x)| : \|x\| = 1 \}
\leq \sup \{ |S(x)| : \|x\| = 1 \} + \sup \{ |T(x)| : \|x\| = 1 \}
= \|S\| + \|T\|
\]
since \( |S(x) + T(x)| \leq |S(x)| + |T(x)| \). Thus, \( \text{hom}_b(V, \mathbb{R}) \) is a normed vector space.

The analytic double dual of a normed vector space is \( \text{hom}_b(\text{hom}_b(V, \mathbb{R}), \mathbb{R}) \). As with the double dual of an arbitrary vector space, we have a natural map \( \eta' : V \to \text{hom}_b(\text{hom}_b(V, \mathbb{R}), \mathbb{R}) \), defined by \( \eta'(v)(f) = f(v) \).

Let \( W \) be a subspace of a vector space \( V \), and let \( T : W \to U \) be a linear transformation. By using bases, we can produce a subspace \( W' \) of \( V \) for which \( V = W \oplus W' \). We can then extend \( T \) to \( V \) by defining \( T(w + w') = T(w) \) for each \( w' \in W' \). This argument shows that we can always extend linear transformations on a subspace to the space itself. The following is the analogue of this result in analysis.
Theorem 6 (Hahn-Banach). Let $V$ be a normed vector space. If $W$ is a subspace of $V$ and $f : W \to \mathbb{R}$ is a bounded linear functional, then there is a linear functional $F : V \to \mathbb{R}$ with $F|_W = f$ and $\|F\| = \|f\|$.

We refer to analysis texts for a proof of this theorem; see, for example, [1, Thm. 5.6]. Its proof uses a Zorn’s lemma argument, as does the abstract vector space analogue we mentioned earlier.

Lemma 7. Let $V$ be a normed vector space, and let $\eta' : V \to \text{hom}_b(\text{hom}_b(V, \mathbb{R}), \mathbb{R})$ be the map defined above by $\eta'(v)(f) = f(v)$. Then $\|\eta'(v)\| = \|v\|$. Thus, $\eta'(v)$ is a bounded linear functional on $\text{hom}_b(V, \mathbb{R})$, and so $\eta'(v) \in \text{hom}_b(\text{hom}_b(V, \mathbb{R}), \mathbb{R})$. Moreover, $\eta'$ is an injective linear transformation.

Proof. It is very easy to prove that $\eta'(v)$ is a linear functional, so the only issue is to prove that it is bounded. We have

$$\|\eta'(v)\| = \sup \{|f(v)| : f \in \text{hom}_b(V, \mathbb{R}), |f| = 1\}.$$ 

Since $|f(v)| \leq \|f\| \cdot |v| = \|v\|$ for $f$ with $\|f\| = 1$, we see that $\|\eta'(v)\| \leq \|v\|$. This is enough to prove that $\eta'(v)$ is bounded. To prove equality, define $f_0 : \mathbb{R}v \to \mathbb{R}$ by $f_0(\alpha v) = |\alpha v|$. It is trivial to see that $f_0$ is a bounded linear functional on $\mathbb{R}v$ with $\|f_0\| = 1$. By the Hahn-Banach theorem, there is a bounded linear functional $f : V \to \mathbb{R}$ such that $f|_{\mathbb{R}v} = f_0$ and $\|f\| = 1$. Since $f(v) = \|v\|$, we see that $\|\eta'(v)\| \geq |f(v)| = \|v\|$. This gives the reverse inequality, and so $\|\eta'(v)\| = \|v\|$.

It is an easy argument to see that $\eta'$ is a linear map. Another application of the Hahn-Banach theorem shows that $\eta'$ is injective: If $v \neq 0$, define $f_0 : \mathbb{R}v \to \mathbb{R}$ by $f_0(\alpha v) = \alpha$. By the Hahn-Banach theorem, there is $f \in \text{hom}_b(V, \mathbb{R})$ with $f(v) = f_0(v) = 1$. Then $\eta'(v)(f) = 1$, so $\eta'(v) \neq 0$. Thus, $\ker(\eta') = \{0\}$, so $\eta'$ is injective.

We now consider the spaces $l^p$, $l^\infty$, and $c_0$. We show how to obtain bounded linear functionals on them in the following lemma.

Lemma 8. Let $x = \{x_n\}, y = \{y_n\}$ be sequences of real numbers. Define $T_y$ by $T_y(x) = \sum_{n=1}^\infty x_n y_n$.

1. Let $y \in l^\infty$. Then $T_y$ is a well-defined linear functional on $l^1$ with $\|T_y\| = \|y\|_\infty$.

2. Let $p, q \geq 1$ with $1/p + 1/q = 1$, and let $y \in l^q$. Then $T_y$ is a well-defined linear functional on $l^p$ with $\|T_y\| = \|y\|_q$.

3. Let $y \in l^1$. Then $T_y$ is a well-defined linear functional on $l^\infty$ with $\|T_y\| = \|y\|_1$. 

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Proof. Once we know that $T_y$ is well-defined; that is, the sequence $\sum_{n=1}^{\infty} x_n y_n$ is convergent for each appropriate $x$, the linearity is easy to prove. For, let $\{x_n\}, \{z_n\}$ be sequences in the appropriate space, and $\alpha, \beta \in \mathbb{R}$. Then

$$
T_y (\alpha \{x_n\} + \beta \{z_n\}) = \sum_{n=1}^{\infty} y_n (\alpha x_n + \beta z_n) = \sum_{n=1}^{\infty} y_n \alpha x_n + \beta y_n z_n
$$

$$
= \alpha \sum_{n=1}^{\infty} y_n x_n + \beta \sum_{n=1}^{\infty} y_n z_n = \alpha T_y (\{x_n\}) + \beta T_y (\{z_n\}).
$$

(1). Let $y \in l^\infty$, and let $x \in l^1$. Then $\sum_{n=1}^{\infty} |x_n y_n| \leq \sum_{n=1}^{\infty} |y|_\infty |x_n| = |y|_\infty \|x\|_1 < \infty$, so $\sum_{n=1}^{\infty} x_n y_n$ is an absolutely convergent series. Thus, $T_y(x) \in \mathbb{R}$, so $T_y$ is well-defined. Moreover, this shows that $\|T\| \leq \|y\|_\infty$. For the reverse inequality, let $e_n$ be the sequence whose $n$-th term is 1 and all other terms 0. Then $e_n \in l^1$ and $T_y(e_n) = y_n$. Thus, $|y_n| = |T(e_n)| \leq \|T\| \|e_n\|_1 = \|T\|$. Thus, $\|y\|_\infty = \sup \{|y_n|\} \leq \|T\|$. Therefore, $\|T\| = \|y\|_\infty$.

(2). We recall the Holder inequality [1, Thm. 3.5], which says that $\sum_{n=1}^{\infty} |x_n y_n| \leq \|x\|_p \|y\|_q < \infty$. Therefore, $T_y$ is well-defined and $\|T\| \leq \|y\|_q$. For the reverse inequality, let $s_N = (\text{sgn}(y_1)|y_1|^{q-1}, \ldots, \text{sgn}(y_n)|y_n|^{q-1}, 0, \ldots) \in l^p$. Then $T(s_N) = \sum_{n=1}^{N} |y_n|^q$. Since $p + q = pq$, the inequality $|T(s_N)| \leq \|T\| \cdot \|s_N\|_p$ says

$$
\sum_{n=1}^{N} |y_n|^q \leq \|T\| \cdot \left( \sum_{n=1}^{N} (|y_n|^{q-1})^p \right)^{1/p} = \|T\| \cdot \left( \sum_{n=1}^{N} |y_n|^q \right)^{1/p} = \|T\| \cdot \left( \sum_{n=1}^{N} |y_n|^q \right)^{1-1/q},
$$

so

$$
\left( \sum_{n=1}^{N} |y_n|^q \right)^{1/q} \leq \|T\|.
$$

Letting $N \to \infty$, we obtain $\|y\|_q \leq \|T\|$, and so $\|T\| = \|y\|_q$.

(3). The argument is virtually identical to that in (1): Let $y \in l^1$, and let $x \in l^\infty$. Then $\sum_{n=1}^{\infty} |x_n y_n| \leq \sum_{n=1}^{\infty} |y_n| \cdot \|x\|_\infty = |y|_1 \cdot \|x\|_\infty < \infty$, so $\sum_{n=1}^{\infty} x_n y_n$ is an absolutely convergent series. Thus, $T_y(x) \in \mathbb{R}$, so $T_y$ is well-defined and $\|T\| \leq \|y\|_1$. For the reverse inequality, let $s_N = (\text{sgn}(y_1), \ldots, \text{sgn}(y_N), 0, \ldots) \in l^\infty$. We have $\|s_N\|_\infty = 1$. Therefore, $\sum_{n=1}^{N} |y_n| = T_y(s_N) \leq \|T\|$. Letting $N \to \infty$, we get $\|y\|_1 \leq \|T\|$. \qed

By restricting the domain, we see that for any $y \in l^1$, the map $T_y$ yields a bounded linear functional $c_0 \to \mathbb{R}$. We will see below that Lemma 8 describes all bounded linear functionals on $l^p$ for all $p \geq 1$ and on $c_0$. To help us do this, recall that a linear transformation is determined by its action on a basis. We need an analogue of this fact for continuous linear transformations. If $V$ is a normed vector space, we call a sequence $\{v_n\}_{n=1}^{\infty}$ of elements of $V$ a topological basis of $V$ if each $x \in V$ can be written in the form $x = \sum_{n=1}^{\infty} a_n v_n$ for some $a_n \in \mathbb{R}$. This means that, for each $x \in V$, there is a sequence of real numbers $\{a_n\}$ such that $\lim_{N \to \infty} \sum_{n=1}^{N} a_n v_n \to x$ with respect to the norm on $V$. The existence of a nice topological basis of $l^p$ for $p \geq 1$ and of $c_0$ will be a key for us in determining their dual spaces.
Lemma 9. Let $e_n$ be the sequence whose $n$-th term is 1 and all of whose other terms are 0. Then $\{e_n\}$ is a topological basis for $l^p$ for each $p \geq 1$ and for $c_0$.

Proof. The $e_n$ are elements of all the sequence spaces we have discussed. Let $x = \{x_n\}$. We claim that $x = \sum_{n=1}^{\infty} x_n e_n$ for all $x$ in the spaces stated in the lemma. We must prove that if $s_N = \sum_{n=1}^{N} x_n e_n$, then $\lim_{N \to \infty} s_N = x$, the convergence taking place in the given space we are considering. That is, we must prove, for each $s \in \alpha, \beta \in \{\}$

Thus, the map $s \rightarrow s_N$ is uniquely determined by the sequence $\{(x_n : n > N) = 0$ since $x_n \rightarrow 0$. Thus, $s_n \rightarrow x$ in $c_0$.

We now determine the analytic dual space $\text{hom}_b(l^p, \mathbb{R})$ and $\text{hom}_b(c_0, R)$. To give some terminology, if $V, W$ are normed vector spaces, then we say that $V \cong W$ as normed spaces if there is a vector space isomorphism $\varphi : V \rightarrow W$ with $||\varphi(v)|| = ||v||$ for all $v \in V$.

Proposition 10.

1. We have $\text{hom}_b(l^1, \mathbb{R}) = \{T_y : y \in l^\infty\}$, and $\text{hom}_b(l^1, \mathbb{R}) \cong l^\infty$ as normed spaces.

2. If $1/p + 1/q = 1$, then $\text{hom}_b(l^p, \mathbb{R}) = \{T_y : y \in l^q\}$, and $\text{hom}_b(l^p, \mathbb{R}) \cong l^q$ as normed spaces.

3. We have $\text{hom}_b(c_0, \mathbb{R}) = \{T_y |_{c_0} : y \in l^1\}$, and $\text{hom}_b(c_0, \mathbb{R}) \cong l^1$ as normed spaces.

Proof. The main idea in all three statements is the following: suppose that $\{v_n\}$ is a topological basis for a normed vector space $V$, and let $T$ be continuous linear functional on $V$. If $x = \sum_{n=1}^{\infty} a_n v_n$, then

$$T(x) = T\left(\lim_{N \to \infty} \sum_{n=1}^{N} a_n v_n\right) = \lim_{N \to \infty} T\left(\sum_{n=1}^{N} a_n v_n\right) = \lim_{N \to \infty} \sum_{n=1}^{N} a_n T(v_n)\quad (2)$$

$$= \sum_{n=1}^{\infty} a_n T(v_n).$$

Thus, $T$ is uniquely determined by the sequence $\{T(v_n)\}$. By Lemma 9, we may work with the topological basis $\{e_n\}$ in all three cases; Equation (2) shows that if $x = \{x_n\}$, then $T(x) = \sum_{n=1}^{\infty} x_n T(e_n)$ for all $x$ in any of the spaces under consideration. Moreover, whenever $T_y$ and $T_z$ are defined, it is a trivial argument to prove that $T_{\alpha y + \beta z} = \alpha T_y + \beta T_z$ for any $\alpha, \beta \in \mathbb{R}$. Thus, the map $y \mapsto T_y$ is linear. Note that if $y = \{y_n\}$, then $T_y(e_n) = y_n$. Thus, if $T_y = 0$, then each $y_n = 0$, so $y = 0$.

1. Let $T \in \text{hom}_b(l^1, \mathbb{R})$. Since $T$ is bounded, $|T(e_n)| \leq ||T|| \cdot ||e_n||$. Thus, $y = \{T(e_n)\} \in l^\infty$, and from the description of $T$ above, we see that $T(x) = \sum_{n=1}^{\infty} x_n T(e_n) = T_y(x)$. Thus,
\[ T = T_y. \] This yields \( \text{hom}_b(l^1, \mathbb{R}) = \{ T_y : y \in l^\infty \}. \) As we pointed out in general, the map \( y \mapsto T_y \) is an injective linear map, and so is an isomorphism. Moreover, since \( \|T_y\| = \|y\|_\infty \), it is an isomorphism of normed spaces.

2. Let \( T \in \text{hom}_b(l^p, \mathbb{R}) \) and set \( y = \{ T(e_n) \}. \) Since \( T \) is bounded, the argument in Statement (2) of Lemma 8 used to prove \( \|y\|_q \leq \|T\| \) shows that \( y \in l^q. \) Thus, as in (1), we obtain the result.

3. Let \( T \in \text{hom}_b(c_0, \mathbb{R}) \) and set \( y = \{ T(e_n) \}. \) The argument in Statement (3) of Lemma 8 shows that \( y \in l^1, \) and as before, we get \( \text{hom}_b(c_0, \mathbb{R}) = \{ T_y : y \in l^1 \}, \) and \( \text{hom}_b(c_0, \mathbb{R}) \cong l^1 \) as normed spaces. \( \square \)

Unlike the case for the spaces \( l^p, \) we have \( \text{hom}_b(l^\infty, \mathbb{R}) \neq \{ T_y : y \in l^1 \}, \) as we will see shortly. The problem is that \( l^\infty \) does not have a topological basis. To help see this, we recall that a topological space \( X \) is said to be separable if \( X \) contains a countable dense subset.

**Proposition 11.** Let \( V \) be a normed vector space. If \( V \) has a topological basis, then \( V \) is separable.

**Proof.** Let \( \{ v_n \} \) be a topological basis for \( V. \) Then \( A := \{ \sum_{n=1}^N q_n v_n : q_n \in \mathbb{Q}, N \geq 1 \} \) is a countable set. We claim that it is dense in \( V. \) To see this, take \( x \in V, \) and write \( x = \sum_{n=1}^\infty a_n v_n \) for some \( a_n \in \mathbb{R}. \) Let \( \varepsilon > 0. \) Then there is an \( N \) such that \( \| x - \sum_{n=1}^N a_n v_n \| < \varepsilon/2. \) Since \( \mathbb{Q} \) is dense in \( \mathbb{R}, \) we can find \( q_n \in \mathbb{Q} \) such that

\[
|a_n - q_n| < \frac{\varepsilon}{2^{n+1} \| v_n \|}
\]

for each \( n. \) Then

\[
\left\| \sum_{n=1}^N a_n v_n - \sum_{n=1}^N q_n v_n \right\| \leq \sum_{n=1}^N |a_n - q_n| \| v_n \| \leq \sum_{n=1}^N \frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{2}.
\]

Consequently, \( \| x - \sum_{n=1}^N q_n v_n \| < \varepsilon. \) This proves that \( A \) is a countable dense subset of \( V. \) Therefore, \( V \) is separable. \( \square \)

As a consequence of the proposition, \( l^p \) for each \( p \geq 1 \) and \( c_0 \) are separable, since we showed in Lemma 9 that each has a topological basis.

**Example 12.** The space \( l^\infty \) is not separable, and so does not have a topological basis; for if \( \{ v_n \}_n \) is a countable subset of \( l^\infty, \) define a sequence \( x = \{ x_n \} \) by

\[
x_n = \begin{cases} 0 & \text{if } v_{n,n} \geq \frac{1}{2} \\
1 & \text{if } v_{n,n} < \frac{1}{2} \end{cases}.
\]

Then \( |x_n - v_{n,n}| \geq 1/2. \) Clearly \( x \in l^\infty \) and \( \| x - v_n \|_\infty \geq 1/2. \) This proves that \( \{ v_n \}_n \) is not dense in \( l^\infty. \)
Proposition 13. The maps $\eta_{l^1} : l^1 \to \text{hom}(l^1, \mathbb{R})$ and $\eta'_{l^1} : l^1 : \text{hom}_b(l^1, \mathbb{R})$ are not surjective.

Proof. An easy calculation shows that $\pi \circ \eta_V = \text{inc} \circ \eta'_V$. The map $\pi$ is surjective since every linear functional on $\text{hom}_b(V, \mathbb{R})$ can be extended to a linear functional on $\text{hom}(V, \mathbb{R})$. If $\eta_V$ were surjective, then $\pi \circ \eta_V$ would be surjective, and this would force $\text{inc}$ to be surjective. We show this is not true. By Proposition 10, we identify $l^\infty$ with $\text{hom}_b(l^1, \mathbb{R})$ by identifying $y$ with $T_y$. Define a linear transformation $S$ on the span of $\{e_n : n \geq 1\}$ by $S(e_n) = n$, and extend $S$ in any way to all of $l^\infty$. Then $S$ is not bounded, so $S$ does lie in the image of $\text{inc}$. Thus, $\eta_{l^1}$ is not surjective.

To see that $\eta'_{l^1}$ is not surjective we need an analytic variant of the argument in the previous paragraph. Note that each $T_y \in \text{hom}_b(l^\infty, \mathbb{R})$ coming from a nonzero element $y \in l^1$ has the property that $T_y$ is a nonzero operator on $c_0$. This is clear since if $y = \{y_n\}$ with $y_m \neq 0$, then $e_m \in c_0$ and $T_y(e_m) = y_m \neq 0$. We will produce a nonzero $T \in \text{hom}_b(l^\infty, \mathbb{R})$ for which $T|_{c_0} = 0$. Let $v$ be the constant sequence whose $n$-th term is 1 for each $n$. Then $v \in l^\infty$ but $v \notin c_0$. The sum $c_0 + Rv$ is direct since clearly the only sequence in $Rv$ converging to 0 is the zero sequence. Consider the linear transformation $T_0 : c_0 + Rv \to \mathbb{R}$ defined by $T_0(w + rv) = r$ for all $r \in \mathbb{R}$ and $w \in c_0$. Then $T_0$ is bounded, since if $\{w_n + r\} \in c_0 + Rv$ has norm 1, then $|w_n + r| \leq 1$ for each $n$. This forces $|r| \leq 1$, and so $|T_0(w + rv)| = |r| \leq 1$. Thus, $\|T_0\| \leq 1$. By the Hahn-Banach theorem, we can extend $T_0$ to a bounded linear functional $T$ on $l^\infty$. Since $T|_{c_0} = 0$, the functional $T \neq \eta'(y)$ for every $y \in l^1$. □

References