This assignment is due at the beginning of class on Friday 9 October. Solve all problems and give complete proofs and explanations. You may use any books you would like, but you must cite your sources. Each group will hand in a single report, which must be typed, though the mathematics may be written by hand.

In the problems below \( F \) is a field and all vector spaces below are \( F \)-vector spaces, and are not assumed finite-dimensional.

1. Let \( U_1, \ldots, U_n \) be subspaces of a vector space \( V \). We call the sum \( U_1 + \cdots + U_n \) direct if each element of the sum has a unique representation as \( u_1 + \cdots + u_n \) with each \( u_i \in U_i \). When the sum is direct, we denote it by \( U_1 \oplus \cdots \oplus U_n \).

   (a) If \( U_1 \cap U_2 = \{0\} \), prove that \( U_1 + U_2 \) is a direct sum.

   (b) Prove that \( U_1 + \cdots + U_n \) is direct if and only if \( U_j \cap \sum_{i \neq j} U_i = \{0\} \) for each \( j = 1 \ldots, n \).

2. Let \( V \) be a vector space.

   (a) Let \( U \) be a subspace of \( V \). Prove that there is a subspace \( W \) of \( V \) with \( V = U \oplus W \).

   (b) Let \( e : V \to V \) be a linear transformation with \( e^2 = e \). Prove that \( V = \ker(T) \oplus \text{im}(T) \).

3. If \( B \) is a basis of a vector space \( V \), prove that \( \text{hom}_F(V,W) \cong \text{map}(B,W) \). Use this to prove that \( V^* \cong \text{map}(B,F) \).

4. Let \( U,W \) be \( F \)-vector spaces and let \( T : U \to W \) be a linear transformation.

   (a) Suppose that \( U \) is a subspace of a vector space \( V \). Prove that there is a linear transformation \( S : V \to W \) with \( S|_U = T \).

   (b) With \( U \) an arbitrary vector space, if \( \sigma : U \to V \) is a 1-1 linear transformation, prove that there is a linear transformation \( S : V \to W \) with \( S \circ \sigma = T \).

   (c) Prove or disprove the previous statement when \( \sigma \) is assumed to be linear, but not necessarily 1-1.
5. Let $V, W$ be vector spaces. If $T \in \text{hom}_F(V, W)$, recall the definition $T^t : W^* \to V^*$ by $T^t(\sigma) = \sigma \circ T$.

(a) if $T : V \to W$ and $S : W \to U$ are both linear, prove that $(S \circ T)^t = T^t \circ S^t$.

(b) Prove that $\ker(T^t) = (\text{im}(T))^0$.

(c) Prove that $\text{im}(T^t) = (\ker(T))^0$. 