1.3 Long Exact Sequences

We can now state and prove perhaps the most fundamental fact of homology, the long exact sequence arising from a short exact sequence of complexes. This long exact sequence will, among other things, help us to compute homology groups.

**Theorem** Let \( 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \) be a short exact sequence of complexes in an Abelian category. Then there are natural maps \( d_n : H_n(C) \to H_{n-1}(A) \) such that

\[
\cdots \to H_{n+1}(C) \xrightarrow{d_{n+1}} H_n(A) \xrightarrow{d_n} H_n(B) \xrightarrow{\partial_n} H_n(C) \xrightarrow{d_n} H_{n-1}(A) \to \cdots \text{ is exact.}
\]

Similarly, if the sequence above is one of cochain complexes, then there is a natural hom \( d^n : H^n(C) \to H^{n-1}(A) \) such that the following is exact

\[
\cdots \to H^{n+1}(C) \xrightarrow{d^{n+1}} H^n(A) \xrightarrow{d^n} H^n(B) \xrightarrow{\partial^n} H^n(C) \xrightarrow{d^n} H^{n-1}(A) \to \cdots
\]

The main tool to prove the theorem is the **snake lemma**.

**Snake Lemma** Given the diagram with exact rows

\[
\begin{array}{ccc}
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \to 0 \\
\downarrow f & & \downarrow g & & \downarrow h & \\
0 & \to A & \xrightarrow{f} & B & \xrightarrow{g} & C
\end{array}
\]

there is an exact sequence

\[
\ker(f') \xrightarrow{\delta} \ker(g) \xrightarrow{\delta} \ker(h) \xrightarrow{\delta} \text{coker}(f') \xrightarrow{\delta} \text{coker}(g) \xrightarrow{\delta} \text{coker}(h).
\]

If \( f' \) is monic, then so is \( \text{ker}(f') \to \text{ker}(g) \), and if \( f \) is monic, so is \( \delta \).
proof we give a (partial) proof in the case of modules: the text on p. 12 indicates how to prove it in general.

The map \( S : \ker(h) \to \text{coker}(f) \) is given as follows. Let \( c \in \ker(h) \), and take \( b \in B' \) with \( f'(b) = c \). Then \( g(b) \in \ker(f) \) by commutativity, so \( g(b) = a(a) \) for some \( a \in A \). We then set \( S(c) = a + \text{im}(f) \in \text{coker}(f) \). This is well defined since if \( b_0 \in B' \) with \( f'(b_0) = c \), then \( \ker(f') \), so \( d - b_0 \in \ker(f) \), so \( c = d + b \in \ker(f) \) for some \( c \in A'. \) If \( g(b_0) = a(a) \), then \( g(b_0 - a) = d' - a \) and \( g(a - b_0) = g(a) - d' = d' \). Injectivity of \( a \) gives \( b_0 - a \in \text{im}(f) \), so \( a + \text{im}(f) = b_0 + \text{im}(f) \in \text{coker}(f) \).

The maps \( \ker(f) \to \ker(g) \) and \( \ker(g) \to \ker(h) \) are restrictions of \( d' \) and \( f' \). The map \( \text{coker}(f) \to \text{coker}(g) \) is given by \( a + \text{im}(f) \mapsto d'(a) + \text{im}(g) \). The map \( \text{coker}(g) \to \text{coker}(h) \) is given similarly.

**Exactness at \( B' \):** The composition \( \ker(f) \to \ker(g) \to \ker(h) \) is clearly 0. Take \( b \in \ker(g) \) with \( f'(b) = 0 \). Then \( \ker(f) = \ker(h) \) for some \( c \in A' \). Also, \( S(c) = g(b) = d' \), so \( f(a) = 0 \) as \( x \) is injective. Thus, \( a \in \ker(f) \).

**Exactness at \( \ker(h) \):** Let \( b \in \ker(h) \). To calculate \( S(f'(b)) \), we take \( b \) as the lift of \( f'(b) \). Since \( g(b) = 0 \), we have \( a = 0 \), so \( \ker(h) \). Next, take \( c \in \ker(h) \) with \( g(c) = 0 \). Write \( c = f'(b) \), \( g(b) = d' h \). Then \( a \in \text{im}(f) \); say \( a = f(b) \). Then \( g(b) = d' h = g(f(b)) \), so \( 0 = b - f(b) \in \ker(g) \). Since \( f'(b) = f'(b) = c \), we are done.

**Exactness at \( \text{coker}(f) \):** Let \( c \in \ker(h) \). Let \( c = f'(b) \), \( g(b) = d' h \) with \( a \in A \). Then \( S(c) = a + \text{im}(f) \). Then \( S(c) = a + \text{im}(f) = g(b) + \text{im}(g) = 0 \). Conversely, \( \ker(f) \to \ker(g) \to \ker(h) \), then \( f(a) = g(b) \) for some \( c \). Let \( a = f(b) \). Then \( \ker(f) \to \ker(g) \to \ker(h) \), then \( f(a) = g(b) \), so \( b_0 - a \in \text{im}(f) \). Then \( a + \text{im}(f) = b_0 + \text{im}(f) \in \text{coker}(f) \).
O map: \( A, B \rightarrow O \), \( O \rightarrow B \) unique maps. Composition is the \( O \) map.

\[
\begin{align*}
A & \xrightarrow{f} B & \xrightarrow{g} C
\end{align*}
\]

\( e \rightarrow \text{im} f \)

\( \text{ker} \)

\( e = \text{gof} = \text{gocoe} \). Since \( e \) is epic, \( \text{goc} = O \). Thus, there is a unique map \( \lambda : \text{im} f \rightarrow \text{ker}g \) with \( \lambda = \text{jod} \). We say the sequence is exact at \( B \) if \( \lambda \) is an isomorphism.

Given \( A \xrightarrow{f} B \), we have

\[
\begin{align*}
A & \xrightarrow{f} B & \xrightarrow{\text{coker}(f)} \text{coker}(f) \\
\uparrow & & \uparrow \\
\text{im} f & = \text{ker}(\text{coker}(f))
\end{align*}
\]

Since \( \text{prof} = 0 \), there is a unique map \( e : A \rightarrow \text{im}(f) \) with \( f = \text{loe} \).

The map \( \lambda \) is monic by definition; \( e \) is epic is not so easy in general.
Thus, \( c \in \ker(h) \), and \( s(c) = a + \text{im}(f) \).

Exactness at \( \text{coker}(g) \): The composition \( \text{coker}(f) \to \text{coker}(g) \to \text{coker}(h) \) is clearly zero. Take \( b + \text{im}(g) \) with \( f(b) = h(c) \) for some \( c \in C' \). Write \( c = \beta'(b') \). Then \( \beta(b - g(b)) = h(c) - \beta(g(b)) = h(c) - h\beta(b) = 0 \). Therefore, we may replace \( b \) by \( b - g(b) \) and assume that \( f(b) = 0 \). Then \( b = d(a) \) for some \( a \). Thus, \( b + \text{im}(g) = \overline{f}(a + \text{im}(f)) \).

If \( \beta' \) is monic, then \( \beta' \ker(f) \) is monic. If \( \beta \) is epic, then clearly \( \overline{f} \) is epic.

Note: \( \overline{f} \) and \( \overline{g} \) are well defined: if \( a + \text{im}(f) = a' + \text{im}(f) \), then \( a - a' = f(e) \) for some \( e \in A' \). Then \( d(a) - d(a') = d(f(e)) = g(d'(e)) \), so \( d(a) + \text{im}(g) = d(a') + \text{im}(g) \).

**Proof of the theorem**

Let \( 0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0 \) be an exact sequence of complexes. From the snake lemma, the following diagram is commutative with exact rows:

\[
\begin{array}{ccccccc}
0 & \to & 0 & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & Z_n(A) & \to & Z_n(B) & \to & Z_n(C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & A_n & \to & B_n & \to & C_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & A_{n-1} & \to & B_{n-1} & \to & C_{n-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{ker}(d) & \to & \text{im}(g) & \to & \text{coker}(f) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
\]
The top and bottom rows come from the snake lemma. Communivity is elementary. Thus, the rows of the following diagram commute.

\[
\begin{array}{ccc}
\mathbb{A}_n & \rightarrow & \mathbb{B}_0 \\
\downarrow d & & \downarrow d \\
\mathbb{C}_n & \rightarrow & \mathbb{O}
\end{array}
\]

\[0 \rightarrow \mathbb{Z}_{n-1}(A) \rightarrow \mathbb{Z}_{n-1}(B) \rightarrow \mathbb{Z}_{n-1}(C)\]

Since \( \ker d = H_n(A) \) and \( \text{coker} d = H_{n-1}(A) \), the snake lemma gives an exact sequence

\[H_n(A) \rightarrow H_n(B) \rightarrow H_n(C) \rightarrow H_{n-1}(A) \rightarrow H_{n-2}(B) \rightarrow H_{n-1}(C)\]

Pasting these together gives the proof.

In the case of modules, we can use the proof of the snake lemma to give a formula for \( S : H_n(C) \rightarrow H_{n-1}(A) \). Let \( z \in H_n(C) \) be represented by \( \mathbf{c} \in \mathbb{C}_n \). Then \( \mathbf{c} = g(\mathbf{b}) \) for some \( \mathbf{b} \in \mathbb{B}_n \). Then \( d(\mathbf{b}) \in \mathbb{B}_{n-1}(B) \). Since we have \( d(\mathbf{c}) = 0 \) by definition, \( g(d(\mathbf{b})) = 0 \), so \( d(\mathbf{b}) = f(\mathbf{a}) \) for some \( \mathbf{a} \in \mathbb{A}_{n-1} \). As \( d(\mathbf{b}) = 0 \) and \( f \) is injective, \( \mathbf{a} \in \mathbb{Z}_{n-1}(A) \), so \( \mathbf{a} \) represents an element \( S(\mathbf{a}) \) in \( H_{n-1}(A) \). A tedious exercise shows this is the correct map.

The existence of the sequence and of the connecting homomorphism \( S \), is not the only important fact. Also important is the "naturalness" of \( S \), which we now explain and prove.
Proposition Let \[ 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \] be a commutative diagram of complexes with exact rows. Then the diagram below commutes:

\[
\begin{array}{cccccccc}
\cdots & H_n(c) & \xrightarrow{d} & H_n(A) & \to & H_n(B) & \to & H_n(C) & \xrightarrow{d} & H_{n-1}(A) & \to & \cdots \\
\downarrow & \quad & \downarrow & & \quad & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & H_n(c') & \xrightarrow{d'} & H_n(A') & \to & H_n(B') & \to & H_n(C') & \xrightarrow{d'} & H_{n-1}(A') & \to & \cdots
\end{array}
\]

Proof We prove this for modules. Since \( H_n \) is a functor, we only need prove this for the squares involving \( \delta \). So, consider the rightmost square above. We use our description of \( S \) above. Take \( c \in Z_n(C) \) representing an element of \( H_n(C) \). Then \( \gamma(c) \cdot c' \) represents its image in \( H_n(C') \). Fix \( s(c) \), write \( c = g(s) \). Then \( s(c') \) is represented by \( d(b) \). Similarly, \( s(c') \) is represented by \( s(d(b)) \), where \( b' = f(b) \), as \( g'(b') \cdot g'(b) = \delta g(b) = \delta(c) \). Now, \( d(s(c)) = d(c) = d(b) = d(b') \), so \( d(s(c)) = s(c') \). This proves commutativity. \( \Box \)

This result can be considered more abstractly, as in p. 14 of the text. There are categories \( \mathcal{L} \) and \( \mathcal{I} \) of short and long exact sequences, and the result above means that we have a functor from \( \mathcal{L} \) to \( \mathcal{I} \).