Two different maps $f,g : A \to B$ between complexes may induce the same maps $H_n(A) \to H_n(B)$ on homology. In this section we discuss the relation $f$ and $g$ must have for this to happen. Note that $f$ and $g$ induce the same maps on homology iff $f-g$ induces the zero map.

The definitions of chain null homotopic and split complex are motivated by topology and the following example:

**Example** complexed of vector space: Let $C$ be a complex of $F$-vector spaces. Set $Z_n = Z_n(C)$, $B_n = B_n(C)$, and $H_n = H_n(C)$. We have short exact sequences

$$0 \to Z_n \to C_n \xrightarrow{d} B_{n-1} \to 0 \quad 0 \to B_n \to Z_n \to H_n \to 0$$

They split, so there are splitting maps $t : B_{n-1} \to C_n$, $u : H_n \to Z_n$, so $C_n = Z_n \oplus t(B_{n-1})$ and $Z_n = B_n \oplus u(H_n)$. Define $s : C_n \to C_{n+1}$ by $C_n \xrightarrow{d} Z_n \xrightarrow{u} B_n \xrightarrow{v} C_{n+1}$. Then

- $d \circ s = d$: if $x = b + t(b') + u(h) \in C_n$, $d(x) = b'$. Also, $d(x) = t(b')$, so $d \circ d(x) = d(t(b')) = b'$. So, $d \circ d = d$.

- $d \circ s$, $s \circ d$ are idempotents with image $B_n$ and $t(B_{n-1})$, respectively.

- $\ker (d \circ s) = H_n$: $s \circ d(n) = s(b') = t(b')$ and $d \circ s(x) = d(t(b')) = b$. So, $(d \circ s)(x) = b + t(b')$, so it is zero iff $b + t(b') = 0$. So, the kernel is $u(H_n) = H_n$.

So, $C$ is split if the existence of $s$ with $d \circ s = d$. The complex is exact iff $H_n = 0$ iff $id_C = d \circ s$.
Definition A complex \( C \) is split if there are maps \( s_n: C_n \to C_{n+1} \) with \( ds + sd = 0 \). If \( C \) is also acyclic, then \( C \) is split exact.

Definition A chain map \( f: C \to D \) is null homotopic if there are maps \( s_n: C_n \to D_{n+1} \) with \( f = ds + sd \). Also, two chain maps \( f, g: C \to D \) are chain homotopic if \( f \sim g \) is null homotopic.

Lemma If \( f \sim g: C \to D \) are chain homotopic, then they induce the same maps \( \tau_n(C) \to \tau_n(D) \).

proof Write \( f = g + ds + sd \) for maps \( s_n: C_n \to D_{n+1} \).
Recall that \( f_\ast: \tau_n(C) \to \tau_n(D) \) is defined by \( f_\ast([\varepsilon]) = f([\varepsilon]) = g([\varepsilon] + \sum_n s_n([\varepsilon]]) \). Now,
\[ (ds + sd)[\varepsilon] = ds(\varepsilon) \in \text{im}(d) = B_{n+1}(D). \]
Thus,
\[ f([\varepsilon]) - g([\varepsilon]) \in B_{n+1}(D), \text{ so } f_\ast([\varepsilon]) = g_\ast([\varepsilon]). \]

we have the following characterizations of these definitions.

Proposition Let \( C \) be a chain complex.
(1) \( C \) is split iff \( C_0 = Z_0 \oplus B_0 \) and \( Z_0 = B_0 \oplus \tau_0'(\text{im} d) \) (\( H_0' \cong H_0 \)).
Furthermore, \( C \) is split exact iff \( H_0' = 0 \).
(2) \( C \) is split exact iff \( \text{id}_C \) is null homotopic.

proof We leave (2) for a handout. One direction of (1) was essentially done in the example above, so suppose \( C \) is split. Then there are maps \( s_n: C_n \to C_{n+1} \) with \( ds + sd = 0 \). We need to produce the splittings.
Since \( ds \circ d = d \), \( ds = id \) on \( \text{im}(d) \). Thus, \( ds = id \) on \( B_{n-1} \). Therefore, if \( t = s|_{B_{n-1}} \), then \( t \) is a splitting map to the sequence \( 0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0 \). Thus, \( C_n = Z_n \oplus t(B_{n-1}) \). Next, consider the sequence \( 0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n \rightarrow 0 \). We have the map \( g : Z_n \rightarrow C_n \rightarrow C_{n+1} \rightarrow B_n \). For the inclusion map \( i : B_n \rightarrow Z_n \), \( \text{d}(c(b)) = c(d(b)) = ds(c(b)) = 0 \). Therefore, \( g \) is a splitting map to the sequence above, so \( Z_n = B_n \oplus H_n \) with \( H_n \cong H_n \). The final conclusion is trivial: \( C \) is exact \( \Rightarrow \) all \( H_n = 0 \) \( \Rightarrow \) all \( H_n' = 0 \). \( \square \)