2.3 Injective Resolutions

In section 2.4, we will use projective resolutions to define the derived functors of a right exact additive functor. However, for left exact functors, we need to dualize everything; that is, we need arrows in the opposite directions. So, we dualize projectivity.

Definition Let \( \mathcal{A} \) be an Abelian category. An object \( I \) is injective if it has the following mapping property: given the following diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & A \\
& \downarrow \phi & \rightarrow \ B \\
& I & \leftarrow
\end{array}
\]

there exists a \( \beta : B \rightarrow I \) with \( \beta \circ \phi = \alpha \).

In other words, maps from a subobject to an injective can be extended to the larger object.

While an injective object is just the dual notion to that of a projective object, examples are not so easy to find. For instance, in \( \text{R-mod} \), free modules are projective. There is a dual notion of cofree, but an exercise shows that the only cofree module is \( 0 \). We will need to work harder to produce examples. We will do so once we record the dual results to those of section 2.3. In particular, we will show that \( \text{R-mod} \) has enough injectives.
We say that \( \mathcal{A} \) has enough injectives if for every object \( A \) there is an injective object \( I \) and a monic \( A \rightarrow I \).

If \( \mathcal{A} \) has enough injectives, then every object has an injective resolution, a complex of the form

\[
0 \rightarrow A \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots
\]

with each \( I^n \) injective, and the complex exact.

We have the following characterization of injectivity. Before we state it, we define, for a category \( \mathcal{A} \), the opposite category \( \mathcal{A}^{\text{op}} \), where the objects are those of \( \mathcal{A} \), and where

\[
\text{hom}_{\mathcal{A}^{\text{op}}} ( A, B ) = \text{hom}_{\mathcal{A}} ( B, A )
\]

**Lemma.** Let \( M \) be an object of an Abelian category \( \mathcal{A} \). Then, the following conditions are equivalent:

1. \( M \) is injective.
2. \( M \) is a projective object of \( \mathcal{A}^{\text{op}} \).
3. The contravariant functor \( \text{hom}_{\mathcal{A}} ( -, M ) \) is exact.

The proof is straightforward. Note that

\[
\text{hom}_{\mathcal{A}} ( -, M ) = \text{hom}_{\mathcal{A}^{\text{op}}} ( M, - )
\]

is a covariant functor whose domain is \( \mathcal{A}^{\text{op}} \). With this technique we can restrict to proving results about covariant functors.
We now state the comparison theorem for injective resolutions.

**Comparison Theorem** Let $M \to I$ and $N \to J$ be injective resolutions. Then any map $f : M \to N$ can be extended to a chain map $f : I \to J$. Furthermore, $f$ is unique up to chain homotopy

$$
\begin{array}{c}
0 \\ f^i \\
\downarrow f^i \\
0 \\
\end{array}
\begin{array}{c}
0 \\ M \\ I^0 \\ I^1 \\ \cdots \\
\downarrow f^i \\
N \\ J^0 \\ J^1 \\ \cdots \\
\end{array}
$$

This theorem follows immediately from the previous comparison applied to the projective resolutions $I^{op} \to M^{op}$, $J^{op} \to N^{op}$ and $f^{op} : N^{op} \to M^{op}$ in $\mathcal{R}^{op}$.

The goal of the rest of this section is to show that $\mathcal{R}$-mod has enough injectives. We do this by first verifying it for $R = \mathbb{Z}$, and then applying some category theory to get it for an arbitrary ring $R$.

The most fundamental fact to help us is the following result, proved by a Zorn's lemma argument.

**Baer's Criterion.** An $R$-module $I$ is injective if and only if for every (left/right) ideal $J$ of $R$, an $R$-module map $J \to I$ can be extended to $R \to I$. 


proof \qquad \text{Given } f: A \to B \text{ injective and } x: A \to I,
\qquad \text{set } \quad L = \{ (b^1, b^2) \mid f^1(b^1) \leq b^2 \leq B, f^1: B^1 \to I, f^1 \circ f = x \}.

and order it by \((b^1, b^2) \leq (b'^1, b'^2)\) if \(b^1 \leq b'^1\) and \(f^2(b^1) = f^2(b'^1)\).

By a standard Zorn argument, \(L\) has a maximal element \((b_0, x_0)\). We will be done once we show \(b_0 = B\).

If not, take \(c \in B - b_0\). Let \(J = \{ r \in R \mid (r,c) \in L \}\). If \(J = \emptyset\), then \(b_0 + Rc = b_0\), so \(b_0 + Rc = b_0 \oplus Rc\), and we can extend \(b_0\) to \(b_0 \oplus Rc\) by sending \(c \to 0\).

So, assume \(J \neq \emptyset\), we have a map
\[ c: J \to Jb_0 \oplus b_0 \to I. \]
This extends to \(R\) by hypotheses. Define \(f^r\) on \(b_0 + Rc\) by \(f^r(b + Rc) = f^r(b) + f^r(c)\).

This is well defined since if \(b + Rc = b' + Sc\), then \((r-s)\) is an \(R\)-equivalence class, so \(r-s \in J\), and
\[ g((r-s)c) = g((r-s)c) = g((b-b')c) = (g(b) - g(b'))(c), \]
so \((g(b) + g(c)) = g(b') + g(c')\). Then \(f^r\) extends \(b_0\), a contradiction to the maximality of \((b_0, x_0)\). Thus, \(b_0 = B\), as desired.

\[ \square \]

Corollary A \(Z\)-module is injective iff it is divisible

proof Recall that \(M\) is divisible if for each \(m \in M\) and \(n \in \mathbb{Z}\), there is an \(m^1 \in M\) with \(n \cdot m^1 = m\).

To see that a divisible group is injective, given \(x: (n) \to M\), set \(m = x(n)\). Find \(m^1\) with \(n \cdot m^1 = m\). Then define \(f: \mathbb{Z} \to M\) by \(f(r) = m^r\), so \(f(r) = rm^1\). This extends \(x\). Conversely, if \(M\) is injective, let \(m \in M\) and \(n \in \mathbb{Z}\). For the map \(x: (n) \to M\), \(x(n) = m\) to extend to \(\mathbb{Z}\), we must have \(m^1 \in M\) with \(n \cdot m^1 = m\), as desired.
From the corollary, we can prove that \(\mathbb{A}b\) has enough injectives.

**Proposition.** \(\mathbb{A}b\) has enough injectives.

**Proof.** Let \(M\) be an Abelian group. The group \(\mathbb{Q}/\mathbb{Z}\) is divisible, so it is injective. Since products of injectives are injective, the group

\[
I = \prod_{\text{f hom}(M, \mathbb{Q}/\mathbb{Z})} \mathbb{Q}/\mathbb{Z} = \left\{ g : \text{hom}(M, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \right\}
\]

is injective. We define \(\alpha : M \to I\) by \(\alpha(m) = f_m\), where \(f_m(\sigma) = \sigma(m)\). Then \(f_m\) is a function from \(\text{hom}(M, \mathbb{Q}/\mathbb{Z})\) to \(\mathbb{Q}/\mathbb{Z}\). It is a group hom since

\[
f_m(\sigma + \tau) = \sigma(m) + \tau(m) = f_m(\sigma) + f_m(\tau).
\]

Finally, we show \(\alpha\) is injective (monic). If \(\alpha\) is bad, then \(f_m = 0\), so \(\sigma(m) = 0\) for all \(\sigma \in \text{hom}(M, \mathbb{Q}/\mathbb{Z})\). If \(\sigma(m) = n > 1\), define \(\sigma : \mathbb{Z}m \to \mathbb{Q}/\mathbb{Z}\) by

\[
m \mapsto n + \mathbb{Z}
\]

and extend to \(M\). If \(\sigma(m) = \infty\), define \(\sigma\) by \(m \mapsto 1/2 + \mathbb{Z}\) and extend to \(M\). This then produces a \(\sigma\) with \(\sigma(m) > 0\), a contradiction unless \(m = 0\). So, \(\alpha : M \to I\) is a monic map into an injective group.

Our remaining goal is to show, using this result and category theory, that \(R\)-mod (or mod-\(R\)) has enough injectives. We need some category theory background to do this.
Definition Let \( \mathcal{A} \) and \( \mathcal{B} \) be categories. A pair of functors \( L: \mathcal{A} \to \mathcal{B} \) and \( R: \mathcal{B} \to \mathcal{A} \) are adjoint (to each other) if there is a natural bijection for all \( A \in \mathcal{A} \), \( B \in \mathcal{B} \)

\[ \tau = \tau_{AB} : \text{hom}_B(L(A), B) \cong \text{hom}_A(A, R(B)). \]

Natural means for all \( f: A \to A' \) in \( \mathcal{A} \) and \( g: B \to B' \) in \( \mathcal{B} \), the following diagram is commutative:

\[
\begin{array}{ccc}
\text{hom}_B(L(A'), B) & \xrightarrow{L(f)^*} & \text{hom}_B(L(A), B) \\
\downarrow \tau & & \downarrow \tau \\
\text{hom}_A(A', R(B')) & \xrightarrow{R(g)^*} & \text{hom}_A(A, R(B))
\end{array}
\]

We call \( L \) the left adjoint and \( R \) the right adjoint of the pair.

Adjoint pairs are common, and we will make use of them often.

Example If \( R \) is a ring and \( M \) is a left \( R \)-module, then \( \otimes_R M : \text{mod-}R \to \text{Ab} \) is left adjoint to \( \text{hom}_{\text{Ab}}(M, -) \). We will prove this in Section 2.6.

Example We will use the following example in this section. Let \( L: \text{mod-}R \to \text{Ab} \) be the forgetful functor; that is, for a module \( M \), \( L(M) \) is the underlying Abelian group \( (M, +) \), forgetting the scalar multiplication. We claim that \( \text{hom}_{\text{Ab}}(R, -) \) is right adjoint to \( L \). Recall that \( \text{hom}_{\text{Ab}}(R, A) \) is a right \( R \)-module via \( (fr)(s) = f(rs) \). To verify this we need to show that this is a natural isomorphism.
\[ \text{hom}_\text{M}(M, A) \cong \text{hom}_\text{R}(\text{M}, \text{hom}(R, A)) \]

We define \( \sigma \) by \( \sigma(f)(m) : f \mapsto f(mr) \). It is easy to see that \( \sigma \) is a group homomorphism. Its inverse \( \tau \) is given by \( \tau(g)(m) = g(mr)(i) \). To see \( \sigma \) and \( \tau \) are inverses, we have, for \( f \in \text{hom}(M, A) \), \( g \in \text{hom}_\text{R}(M, \text{hom}(R, A)) \)

\[ \sigma \tau(g)(m)(r) = \tau(g)(mr) = g(mr)(i) \]

and

\[ g(m)(r) = g(m)(ri) = [g(m)](i) \text{. But, } g(mr) = g(mr) \text{ since } g \text{ is an } R\text{-hom. So, } \sigma \tau(g) = g. \]

Also,

\[ \tau \sigma(f)(m) = \sigma(f)(m)(i) = f(m) \text{. So, } \tau \sigma(f) = f. \]

we will use this pair of functors with the following results.

**Proposition** Suppose \( L : A \rightarrow B \) and \( R : B \rightarrow A \) are an adjoint pair, \( L \) exact, and \( R \) additive. If \( I \) is an injective object in \( B \), then \( RL(I) \) is an injective object in \( A \).

**proof** This is just chasing definitions. Roughly, since

\[ \text{hom}_A(-, RL(I)) \cong \text{hom}_B(-, I) \text{, } \text{hom}_B(-, RL(I)) \text{ is exact since } I \text{ is injective, so } RL(I) \text{ is injective. We know } \text{hom}_A(-, N) \text{ is left exact for any } N \text{. We have the following commutative diagram, for } 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \text{ exact:} \]

\[ 0 \rightarrow \text{hom}_A(C, RL(I)) \rightarrow \text{hom}_A(B, RL(I)) \rightarrow \text{hom}_A(A, RL(I)) \]

\[ 0 \rightarrow \text{hom}_B(CL, I) \rightarrow \text{hom}_B(L(B), I) \rightarrow \text{hom}_B(L(A), I) \rightarrow 0 \]
with the bottom row exact, chasing shows that the top sequence is exact, so \( R(I) \) is injective. \( \square \)

**Corollary** If \( I \) is an injective abelian group, then \( \text{hom}(R,I) \) is injective in \( \text{mod}-R \).

We can now show that \( \text{mod}-R \) has enough injectives.

**Proposition** \( \text{mod}-R \) has enough injectives.

**Proof** Let \( M \) be a right \( R \)-module. Viewing \( M \) as an Abelian group, there is an injection \( \phi: M \to I \) into an injective group \( I \). Then \( \text{hom}(R,I) \) is an injective \( R \)-module by the corollary. We have a map \( M \to \text{hom}(R,I) \) as the following composition:

\[
M \to \text{hom}_R(R,M) \to \text{hom}_R(R,M) \to \text{hom}(R,I)
\]

\[
m \mapsto r \mapsto rm \quad \circ \quad \text{hom}(R,I)
\]

The first map is clearly injective (and an isomorphism) \((\text{if } I \in R)\). The last map is injective, since if \( \alpha \circ \sigma = 0 \), then \( \sigma = 0 \) since \( R \) is monic.

So, we have produced an injective \( R \)-module \( \text{hom}(R,I) \) and a monic \( \phi: M \to \text{hom}(R,I) \).

Thus, \( \text{mod}-R \) has enough injectives. \( \square \)