2.5 Right Derived Functors

Let $T$ be a left exact additive functor $T: \mathcal{A} \to \mathcal{B}$. Given an object $M$ of $\mathcal{A}$, we choose an injective resolution $\varepsilon: M \to I$ (we assume $\mathcal{A}$ has enough injectives) and define the right derived functor $R^nT$ by $R^nT(M) = H^n(T(I))$. The results of the last section will dualize to this situation, since we may view $T^\text{op}: \mathcal{A}^\text{op} \to \mathcal{B}^\text{op}$, a covariant right exact functor. We have $\text{LH}T^\text{op}: \mathcal{A}^\text{op} \to \mathcal{B}^\text{op}$ for each $n$, so $(\text{LH}T^\text{op})^\text{op}: \mathcal{A} \to \mathcal{B}$, and $R^nT(M) = (\text{LH}T^\text{op})^\text{op}(M)$. By this idea we immediately get all the results of 2.4 for right derived functors.

Example. Let $R$ be a ring and $A$ an $R$-module. If $T = \text{Hom}_R(A, -)$, then we denote the right derived functors by $R^nT(B) = \text{Ext}^n_R(A, B)$. This is defined since $T$ is left exact and additive, and $R$-modules has enough injectives. We then have $\text{Ext}^0_R(A, B) = \text{Hom}_R(A, B)$ and given a short exact sequence $0 \to M \to N \to P \to 0$ of $R$-modules, the long exact sequence is

$$0 \to \text{Hom}_R(A, M) \to \text{Hom}_R(A, N) \to \text{Hom}_R(A, P) \to \text{Ext}^1_R(A, M) \to \ldots$$

More generally, if $\mathcal{A}$ is an Abelian category, we can define $\text{Ext}_\mathcal{A}^n(A, B) = R^n(\text{Hom}_\mathcal{A}^\text{op}(A, -))(B)$, provided that $\mathcal{A}$ has enough injectives.

One consequence of the definition is that, for any injective $I$, $R^nT(I) = 0$ for $n \geq 1$. This is because $0 \to I \to I \to 0$ is an injective resolution of $I$. 
We comment about indices in cohomology. Given an injective resolution $0 \to M \to I^0 \to I^1 \to \cdots$, applying $T$ gives $0 \to T(I^0) \to T(I^1) \to \cdots$, and $R^nT(M) = H^n(T(I)) = \ker d^n/\text{im} d^{n+1}$. Since $T$ is left exact, $0 \to M \to I^0 \to I^1$ exact yields $0 \to T(M) \to T(I^0) \to T(I^1)$ is exact, so $\ker d^n = T(M)$. Thus, $R^nT(M) = \ker d^n = T(M)$.

We have assumed our functors are covariant. If $T: A \to B$ is contravariant and left exact, then we view $T: A^{\text{op}} \to B$, a covariant left exact functor. Some injective resolutions in $A^{\text{op}}$ are projective resolutions in $A$, to calculate $R^nT$, given $M$ we produce a projective resolution $P \to M$. Then $R^nT(M) = H^n(T(P))$. So, we have $T(P_0) \to T(P_1) \to T(P_2) \to \cdots$, and $R^nT(M)$ gives the homology of this sequence.

In particular, if $T$ is contravariant and left exact, if $P$ is projective in $A$, then $P$ is injective in $A^{\text{op}}$, so $R^nT(P) = 0$ for $n \geq 1$.

We also comment on why we use injective resolution for a covariant left exact $T$. If, instead we had a projective resolution $P \to M$, exactness of $P_0 \to P_1 \to P_2 \to \cdots$, only gives $T(P_0) \to T(P_1) \to T(M)$. Then $R^nT(M) = \ker d^n/\text{im} d^{n+1} \subseteq T(M)$, but not in general, so $R^nT \neq T$.

To illustrate the use of the long exact sequence, we would Exercise 2.5.1. Exercise 2.5.2 is very much the same.
Proposition. The following conditions are equivalent:

1. \( B \) is injective;
2. \( \hom(-, B) \) is exact;
3. \( \Ext^n_\mathcal{A}(A, B) = 0 \) for all \( n \neq 1 \), all \( A \in \mathcal{A} \);
4. \( \Ext^1_\mathcal{A}(A, B) = 0 \) for all \( n \neq 1 \).

Proof. We already know that (1) \( \Rightarrow \) (2). The condition (3) \( \Rightarrow \) (4) is obvious. For (1) \( \Rightarrow \) (3), let \( A \) be an object. Since \( \Ext^n_\mathcal{A}(A, B) = \mathcal{R}^n(\hom_\mathcal{A}(A, -))(B) \), this is 0 for all \( n \neq 1 \) since \( B \) is injective. Finally, for (4) \( \Rightarrow \) (1), suppose \( 0 \to M \to N \to P \to 0 \) is exact. The long exact sequence (for \( T = \hom_\mathcal{A}(-, B) \)) is

\[
0 \to \hom_\mathcal{A}(P, B) \xrightarrow{\partial} \hom_\mathcal{A}(N, B) \xrightarrow{\partial} \hom_\mathcal{A}(M, B) \xrightarrow{\partial} \Ext^1_\mathcal{A}(P, B) \to \ldots
\]

Since \( \Ext^1_\mathcal{A}(P, B) = 0 \), the sequence for hom is exact. Thus, \( \hom_\mathcal{A}(-, B) \) is exact, so \( B \) is injective. \( \square \)

There is an obvious question that arises from the fact that \( \hom \) has two variables. If \( T = \hom_\mathcal{A}(A, -) \) and \( S = \hom_\mathcal{A}(-, B) \), is \( \mathcal{R}^nT(B) = \mathcal{R}^nS(A) \)? In other words, is

\[
\mathcal{R}^n(\hom(A, -))(B) = \mathcal{R}^n(\hom(-, B))(A) ?
\]

The answer is yes. This is proved in Section 2.7. However, we will give a proof of this in Chapter 5 as an application of spectral sequences. In fact, deep down the proof in Section 2.7 is an ad-hoc argument using spectral sequence ideas.