3.4 Ext and Extensions

In this section we show why Ext has its name by relating $\text{Ext}^1(A,B)$ to extensions of modules. We will see a similar (but different) connection when we study group cohomology (the function $\text{Ext}^1(G,Z)$) in Chapter 6.

An extension of $A$ by $B$ is an exact sequence $0 \to B \to X \to A \to 0$. This definition is valid in any Abelian category. However, we will only consider $R$-mod or mod-$R$.

**Definition.** Two extensions $0 \to B \to X \to A \to 0$ and $0 \to B \to Y \to A \to 0$ are equivalent if there is a commutative diagram

\[
\begin{array}{ccc}
0 & \to & B \\
\downarrow & & \downarrow \\
X & \to & A \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

with $Y$ an isomorphism. ☐

We have previously seen this definition in discussing split exact sequences.

Given $A$ and $B$, there always is an extension of $A$ by $B$, namely the split extension

\[
0 \to B \to A \oplus B \xrightarrow{\pi} A \to 0.
\]

The question we address is how can we classify extensions, and in particular, how can we determine when an extension is split?
It is clear that "equivalence" is an equivalence relation, so we will consider equivalence classes of extensions.

Example  Consider $R = \mathbb{Z}$ and $A = B = \mathbb{Z}_p$ for some prime $p$. We have the split extension $0 \to \mathbb{Z}_p \to \mathbb{Z}_p \oplus \mathbb{Z}_p \to \mathbb{Z}_p \to 0$. Note that any extension of the form $0 \to \mathbb{Z}_p \to \mathbb{Z}_p \oplus \mathbb{Z}_p \to \mathbb{Z}_p \to 0$ is split since the terms are $\mathbb{Z}_p$-vector spaces, so by using bases we can produce splitting maps. What other extensions do we have? The "middle" group in such an extension is an Abelian group of order $p^2$, so it is (up to isomorphism) $\mathbb{Z}_p \oplus \mathbb{Z}_p$ or $\mathbb{Z}_p^2$. We can produce several extensions using $\mathbb{Z}_p^2$. Let $\pi : \mathbb{Z}_p^2 \to \mathbb{Z}_p$ be the surjection

$$\pi(n + p\mathbb{Z}) = n + p\mathbb{Z}.$$  

For $1 \leq m \leq p-1$, define

$$0 \to \mathbb{Z}_p \overset{\alpha}{\to} \mathbb{Z}_p^2 \overset{\gamma}{\to} \mathbb{Z}_p \to 0 \text{ with } (E_c)$$

$$\alpha(n + p\mathbb{Z}) = n + p\mathbb{Z},$$

It is clear that $(E_c)$ is an extension. Fixing $\pi$, any such exact sequence must be one of these since $\mathbb{Z}_p \to \langle p + p\mathbb{Z} \rangle$ in order to have exactness.

We leave as an exercise that changing $\pi$ will only lead to extensions equivalent to one of these. We show no two $E_c$ are equivalent. This will show that there are exactly $p$ equivalence classes of $\mathbb{Z}_p$ by $\mathbb{Z}_p$. Later on we will see how this reflects the fact that $\text{Ext}^1(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$.

Suppose $\psi : \mathbb{Z}_p^2 \to \mathbb{Z}_p^2$ is an isomorphism, and consider the following diagram:
\[
0 \to \mathbb{Z}_p \xrightarrow{\cdot p} \mathbb{Z}_p^2 \xrightarrow{\text{id}} \mathbb{Z}_p \to 0 \\
0 \to \mathbb{Z}_p \xrightarrow{f} \mathbb{Z}_p^2 \xrightarrow{\text{id}} \mathbb{Z}_p \to 0
\]

Suppose the diagram is commutative. The second square then says \( p(1+p^2) = p(x_1 + p^2) \), or
\[1 + p^2 = x_1 + p^2.\] Thus, \( x_1 \equiv 1 \mod p \). Looking at the first square,
\[ dy(\tilde{r}) = r(dx(\tilde{r})), \] or \[ f^p + p^2 = df^p(\tilde{r} + p^2) \]. Thus,
\[ f^p \equiv f^p x_1 \mod p^2, \] or \[ f^p \equiv 1 \mod p. \] Since \( x_1 \equiv 1 \mod p \), we get \( f = 1 \mod p \), so \( f = 1 \).

**Example** We give another example of extensions, although this one is somewhat trivial. Let \( P \) be a projective \( R \)-module, and consider an extension of the form

\[ 0 \to A \to B \to P \to 0. \]

Since \( P \) is projective, this sequence splits. Thus, there is only one equivalence class of extensions of \( P \) by \( A \). This will reflect the fact that \( \text{Ext}^1_R(P, A) = 0 \). Similarly, if \( I \) is injective, then any extension of the form

\[ 0 \to I \to A \to B \to 0 \]

is split, which reflects \( \text{Ext}^1_R(B, I) = 0 \).

We now make a more precise connection between \( \text{Ext} \) and extensions. The idea is to define an element of \( \text{Ext}^1_R(A, B) \) that determines when \( 0 \to B \to X \to A \to 0 \) is split.
Definition Let $0 \to B \xrightarrow{m} X \xrightarrow{n} A \to 0$ be an extension, denote it by $E$. We define $\Theta(E) \in \text{Ext}_R^1(A, B)$ as follows.

Considering $\text{Ext}_R^1(-, B)$, part of the long exact sequence is

$$\text{hom}_R(X, B) \xrightarrow{s} \text{hom}_R(B, B) \xrightarrow{s'} \text{Ext}_R^1(A, B)$$

We set $\Theta(E) = s(id_B)$.

Our first task is to show that $\Theta(E)$ only depends on the equivalence class of $E$. To see this, suppose we have two equivalent extensions

$$0 \to B \xrightarrow{m} X \xrightarrow{n} A \to 0 \quad E$$

$$0 \to B' \xrightarrow{m'} X' \xrightarrow{n'} A' \to 0 \quad E'$$

By the naturality of $s$, we have the commutative diagram

$$\begin{array}{ccc}
\text{hom}_R(X, B) & \xrightarrow{s} & \text{hom}_R(B, B) \\
\downarrow s & & \downarrow s' \\
\text{hom}_R(X', B) & \xrightarrow{s'} & \text{hom}_R(B, B) \\
\end{array}$$

The maps between $\text{hom}_R(B, B)$ are the identity since $\text{hom}_R(-, B)$ is a functor. The maps between $\text{Ext}_R^1(A, B)$ are also the identity since they arise from the functor $\text{Ext}_R^1(-, B)$ acting on $id_A$. From this diagram it is then clear that

$$\Theta(E) = s(id_B) = s'(id_B) = \Theta(E').$$
A common use of homological algebra is to define an element of a homology or cohomology group and see that it is an "obstruction" to certain nice behavior occurring. We now show that $\Theta(E)$ is the obstruction to $E$ being split.

**Lemma** Let $0 \to B \xrightarrow{\epsilon} X \xrightarrow{\tau} A \to 0$ be a short exact sequence of modules. Then $E$ is split if and only if $\Theta(E) = 0$.

**Proof** Suppose that $\Theta(E) = 0$. From the sequence

$$\hom_R(X,B) \xrightarrow{\epsilon^*} \hom_R(B,B) \xrightarrow{\delta} \Ext^1_R(A,B)$$

and $\Theta(E) = \delta(\Id_B)$, we see that $\Id_B \in \ker \delta = \text{im} \epsilon^*$. Thus, $\Id_B = \epsilon^*(\sigma)$ for some $\sigma \in \hom_R(X,B)$. So, $\sigma : X \to B$ is a splitting of $E$.

$$0 \to B \xrightarrow{\epsilon} X \xrightarrow{\tau} A \to 0$$

so $E$ is split.

Conversely, suppose $E$ is split. Then there is a $\sigma \in \hom_R(X,B)$ with $\sigma \circ \tau = \Id_B$. So, $\epsilon^*(\sigma) = \Id_B$. Thus,

$$\Theta(E) = \delta(\Id_B) = \delta(\epsilon^*(\sigma)) = (\delta \circ \epsilon^*)(\sigma) = 0.$$

**Corollary** If $\Ext^1_R(A,B) = 0$, then every extension of $A$ by $B$ is split.

**Proof** If $\Theta(E) = \Ext^1_R(A,B) = 0$, and if $0 \to B \xrightarrow{\epsilon} X \xrightarrow{\tau} A \to 0$ is an extension, then $\Theta(E) \in \Ext^1_R(A,B) = 0$, so $E$ is split by the lemma.
We now show that \( \Theta(E) \) determines when an extension \( E \) is split (e.g., trivial). However, we will show much more, that \( \text{Ext}_k(A, B) \) "classifies" all extensions of \( A \) by \( B \). For, we show that \( \Theta \) is a bijection between equivalence classes of extensions of \( A \) by \( B \) and \( \text{Ext}_k(A, B) \). Moreover, there is an operation on extensions that turns the set of equivalence classes into a group, and when done so, \( \Theta \) is a group isomorphism.

To aid in the proof, we give the description of the operation first. This construction and the proof to follow require the use of pullbacks and pushouts.

**Beef Sum**

Let \( 0 \to B \xrightarrow{\beta} X \xrightarrow{\gamma} A \to 0 \) and \( 0 \to B \xrightarrow{\beta'} Y \xrightarrow{\gamma'} A \to 0 \) be extensions of \( A \) by \( B \).

We refer to these as \( E \) and \( E' \), respectively. We produce an extension that we will call \( E + E' \), the beef sum of \( E \) and \( E' \).

Let \( U \) be the pullback of \( \gamma, \gamma' \).

That is,
\[
U = \{(x, y) \mid \gamma(x) = \gamma'(y) \} \subseteq X \times Y
\]

We then have two injections of \( B \) into \( U \),
\[
b \mapsto (\beta(b), 0) \quad \text{and} \quad b \mapsto (0, \beta'(b)).
\]
We also have a surjection \( U \to A, (x, y) \mapsto \gamma(x) \left( = \gamma'(y) \right). \)

This map has kernel \( \{(\beta(b), \beta'(b)) \mid b, b' \in B \} \), the sum of the two copies of \( B \) via \( \beta \) and \( \beta' \). If we set
\[
V = \{(\beta(b), -\beta'(b)) \mid b, b' \in B \} \quad \text{and} \quad Z = U/U,
\]
then \( V \to 0 \): under the map \( U \to A \), so we have an induced map \( Z \to A \), given by
\[
x(Z((x, y))) = \gamma(x) - \gamma'(y).
\]
The kernel of $r''$ is $\{(g(b),0) \mid b \in B \} = \{(0,d'(b)) \mid b \in B \}$ (since $g(b),0) \equiv (0,d'(b)) \mod \mathbb{U}$). If we define $d'' : E \rightarrow \mathbb{Z}$ by $d''(b) = (g(b),0) + \mathbb{U}$, then
\[
0 \rightarrow B \xrightarrow{d''} \mathbb{Z} \xrightarrow{r''} A \rightarrow 0
\]
is an exact sequence. It is this that we will call the \textit{Baer sum} $E + E'$ of $E$ and $E'$.

We first show that this sum is compatible with equivalence. Let $E$ and $E'$, and $E''$, $E'''$ be equivalent extensions:
\[
\begin{align*}
E & : 0 \rightarrow B \xrightarrow{d} X \xrightarrow{r} A \rightarrow 0 \\
E' & : 0 \rightarrow B \xrightarrow{d'} X' \xrightarrow{r'} A \rightarrow 0 \\
E'' & : 0 \rightarrow B \xrightarrow{d''} X'' \xrightarrow{r''} A \rightarrow 0 \\
E''' & : 0 \rightarrow B \xrightarrow{d'''} X''' \xrightarrow{r'''} A \rightarrow 0
\end{align*}
\]
So, $\phi$, $\psi$ are isomorphisms, and the diagrams commute. Let $E + E'$, $E'' + E'''$ be the corresponding Baer sums.
\[
\begin{align*}
0 & \rightarrow B \xrightarrow{d''} \mathbb{Z} \xrightarrow{r''} A \rightarrow 0 \\
0 & \rightarrow B \xrightarrow{l''} \mathbb{Z}' \xrightarrow{r''} A \rightarrow 0
\end{align*}
\]
Define $\phi : \mathbb{Z} \rightarrow \mathbb{Z}'$ by $\phi((x,y)) = (\phi(x), \phi'(y))$. This is well defined since we have a group homomorphism $X \times Y \rightarrow \mathbb{Z}'$, $(x,y) \mapsto (\phi(x), \phi'(y))$ (as $X \times Y = X \otimes Y$), and $\phi((x,y)) = (\phi(x), \phi'(y))$, or the image is in $\mathbb{Z}'$.

Moreover, $(g(b),-g'(b)) \mapsto (\phi(g(b))-\phi'(g'(b))) = (g(b),-g'(b)) = 0$.

So, $\phi$ is a homomorphism. It is an isomorphism since its inverse is given by $(x,y) \mapsto (\phi^{-1}(x), \phi'^{-1}(y))$.

Moreover, the following diagram commutes.
\[ 0 \to B \xrightarrow{\phi} A \to 0 \]
\[ 0 \to B \xrightarrow{\psi} A \to 0 \]

To see this, \( \phi(g(b)c) = \psi(\phi(b)c) = (\phi(\phi(b)c)) = (c(\phi(b)c)) = c(b) \)
and
\[ \sigma(\phi(\phi(b)c)) = \sigma(c(\phi(b)c)) = \sigma(c(\phi(b)c)) = \sigma(c(b)) = \tau(c(b)) = \tau(\phi(b)c) \).

We are now in a position to prove that \( \text{Ext}^1_k(A,B) \)
classified extensions.

**Theorem**: The map \( \Theta \) is a bijection between the set of
equivalence classes of extensions of \( A \) by \( B \) and \( \text{Ext}^1_k(A,B) \). Moreover, with Baer sum, the eg classes
form an Abelian group, and \( \Theta \) is a group homomorphism.

**proof**: We first show \( \Theta \) is a bijection. Let
\[ \begin{align*}
E_1 &\cong A \\
E_2 &\cong B \\
E_3 &\cong \text{Ext}^1_k(A,B)
\end{align*} \]
be extensions.

Let \( \xi \in \text{Ext}^1_k(A,B) \). We need to construct an extension \( E \) with \( \Theta(E) = \xi \). First, pick a projecting
module mapping onto \( A \), and let
\[ 0 \to M \xrightarrow{P} A \to 0 \]
be the resulting exact sequence (\( M = \ker P \)).
The long exact sequence for $\text{Ext}_R^*(-, B)$ yields

$$\text{hom}_R(M, B) \xrightarrow{\delta} \text{hom}_R(M, B) \xrightarrow{\delta} \text{Ext}_R^1(A, B) \xrightarrow{\delta} \text{Ext}_R^1(P, B) = 0,$$

the latter term is $0$ since $P$ is projective. Therefore, $\delta$ is onto. Take $P : M \to B$ with $\delta(p) = x$. Next, form the pushout of

$$
\begin{array}{ccc}
M & \xrightarrow{p} & P \\
\downarrow & & \downarrow \\
B & \xrightarrow{\delta} & X
\end{array}
$$

and the diagram

$$
\begin{array}{ccc}
0 & \to & M & \xrightarrow{p} & P & \xrightarrow{\delta} & A & \to & 0 \\
& & \downarrow{\beta} & & \downarrow{\beta} & & \downarrow{\beta} & & \\
0 & \to & B & \xrightarrow{\delta} & X & \xrightarrow{\delta} & A & \to & 0
\end{array}
$$

The map $\tau : X \to A$ is defined by $\tau((p, b)) = \sigma(p)$. This is well defined since $\text{Ext}_R^1(M, -) \to \sigma(M) = 0$. It is clear that the right square is commutative. The bottom row is exact by the handout. (not in handout) So, applying the naturality of $\delta$ to this diagram gives the commutative square

$$
\begin{array}{ccc}
\text{hom}_R(M, B) & \xrightarrow{\delta} & \text{Ext}_R^1(A, B) \\
\uparrow{\delta^*} & & \uparrow{\delta^*} \\
\text{hom}_R(B, B) & \xrightarrow{\delta} & \text{Ext}_R^1(A, B)
\end{array}
$$

The map on $\text{ext}$ groups is $1_B$, since it is $\text{Ext}_R^1(-, B)(1_B)$ and $\text{Ext}_R^l(-, B)$ is a functor. So,

$$
\Theta(E) = \delta(1_B) \text{ and } \delta'(\phi) = x, \text{ but } \delta^*(1_B) = B, \text{ so commutativity gives } \Theta(E) = x.
$$
We will use the fixed extension $0 \to B \to P \to A \to 0$ in the rest of the argument. Note that we can calculate $\theta(E)$ as $s(p)$ for any $p : M \to B$ satisfying $s(p) = \theta(E)$.

To see that $\theta(E + E') = \theta(E) + \theta(E')$, let

$E : 0 \to B \to X \to A \to 0$
$E' : 0 \to B' \to Y \to A' \to 0$

be two extensions. Then $p \circ \pi : M \to B$ with $s(p \circ \pi) = s(p) = \theta(E)$. Following the argument of surjectivity above, we have maps

we have a map $\tilde{s} : P \to X$ with $\tau \circ \tilde{s} = s$ by projection:

So, we have a diagram

$0 \to M \to P \to A \to 0$
$0 \to B \to X \to A \to 0$

where, viewing $M = P$, $p = \alpha p$. So, this diagram commutes, and the argument for the surjectivity of $\theta$ shows $s(p) = \theta(E)$. Similarly, there is a commutative diagram.
\[ 0 \to M \xrightarrow{\iota} P \xrightarrow{\epsilon} A \to 0 \]
\[ 0 \to B \xrightarrow{d'} \gamma \xrightarrow{\delta} A \to 0 \]

with \( \delta(\gamma') = \Theta(\epsilon'). \)

We let \( 0 \to B \xrightarrow{d''} Z \xrightarrow{r''} A \to 0 \) be the Bass sum \( E + E', \) we have a map \( \sigma'' : P \to Z \) given by:

\[
\sigma''(p) = (d(p), d'(p))
\]

This is well defined since \( \tau d(p) = r'd'(p) = \sigma(p) \)

The restriction to \( M \) is \( \beta + \beta' \) since:

\[
\sigma''(\iota(m)) = (\iota \iota(m), \sigma'(m)) = (\delta \beta(m), \delta' \beta'(m))
\]

\[
= (\delta \beta(m), 0) + (0, \delta' \beta'(m))
\]

\[
= \delta''(\beta(m)) + \delta''(\beta'(m)) = \delta''((\beta + \beta')(m)).
\]

So, we have a commutative diagram:

\[ 0 \to \gamma \xrightarrow{\iota} P \xrightarrow{\epsilon} A \to 0 \]

\[ 0 \to B \xrightarrow{d''} Z \xrightarrow{\delta} A \to 0 \]

And so, \( \Theta(E + E') = \delta(\beta + \beta') = \delta(\beta) + \delta(\beta') \)

\[ = \Theta(E) + \Theta(E'). \]

We leave it as an exercise to see that this means \( \Theta \) is a group isomorphism. Note: the lemma gives injectivity.
Argument variation: without Baer sum, why is $\Theta$ injective.

We have shown that $\Theta$ is surjective. We wish to show it is injective. So, suppose that $\Theta(E) = \Theta(E')$, where

$E: 0 \to B \xrightarrow{g} X \xrightarrow{r} A \to 0$

$E': 0 \to B \xrightarrow{g'} X' \xrightarrow{r'} A \to 0$

Since $P$ is projective there is a hom $\xi: P \to X$ (resp. $\xi': P \to X'$) with $r\xi = 0$ (resp. $r'\xi' = 0$)

\[ \begin{array}{ccc}
P \xrightarrow{\xi} X \\
\downarrow \quad \downarrow \xi' \\
A \xrightarrow{r} 0
\end{array} \]

So, we have the following commutative diagrams

\[ \begin{array}{ccc}
0 & \to & M \xrightarrow{\phi} P \\
\downarrow \phi' & \searrow & \downarrow \phi \\
0 & \to & B \xrightarrow{g} X \xrightarrow{r} A \to 0
\end{array} \]

\[ \begin{array}{ccc}
0 & \to & M \xrightarrow{\phi'} P' \\
\downarrow \phi''' & \searrow & \downarrow \phi' \\
0 & \to & B \xrightarrow{g'} X' \xrightarrow{r'} A \to 0
\end{array} \]

where $\phi = \phi'M$ and $\phi' = \phi'M$. We wish to show that $E \cong E'$. We have

\[ \begin{array}{ccc}
0 & \to & B \xrightarrow{g} X \xrightarrow{r} A \to 0 \\
\downarrow & \downarrow \\
0 & \to & B \xrightarrow{g'} X' \xrightarrow{r'} A \to 0
\end{array} \]

and we need $\Phi: X \cong X'$ to make the diagram commute. We will use the ORM for pushouts. By the handout, $X$ is the pushout of $\phi$ and $\phi'$. Similarly, $X'$ is the pushout of $\phi'$ and $\phi'$. 
To use the map for \( X \), we need a map \( \beta \Rightarrow X' \). The map \( \beta' \) does not work since we need \( \beta' \phi = \phi' \), which is false. We have \( S(\beta) = S(\beta') \) since \( \Theta(E) = \Theta(E') \) and from the surjectivity argument. So, \( \beta' - \beta \in \ker S = \text{Im} \phi' \), so \( \beta' = \beta + f \circ \lambda \) for some \( f : P \Rightarrow B \). Then

\[
\phi' \beta = \phi'(\beta' - \beta \phi) = \phi' \beta' - \phi' \beta \phi = \phi' \lambda - \phi' \phi \lambda = (\phi' - \phi') \lambda
\]

So, with \( \phi' - \phi' \lambda : P \Rightarrow X' \) and \( \phi' : B \Rightarrow X' \), there is a unique \( \psi : X \Rightarrow X' \) with

\[
\psi \beta = \phi' \lambda \quad \text{and} \quad \psi \phi = \phi' - \phi' \lambda
\]

we thus have commutativity of

\[
\begin{array}{ccc}
B & \xrightarrow{j} & X' \\
\downarrow \phi' & & \downarrow \psi \\
B & \xrightarrow{=} & X'
\end{array}
\]

We also need

\[
\begin{array}{ccc}
X & \xrightarrow{f} & A \\
\downarrow \psi & & \downarrow \phi \\
X' & \xrightarrow{=} & A
\end{array}
\]

which means \( \tau' \psi = \tau \). To see this, recall from the proof of Prop. 2 of the handout that \( X = \phi(P) + \phi(B) \).
If we show \( (\tau' \psi - \tau') \alpha = 0 = (\tau' \psi - \tau) \alpha \), then \( \tau' \psi = \tau \).
To see these are true,

\[(\gamma^\prime \psi - \gamma) \phi = \gamma^\prime \phi \gamma - \gamma \phi = \gamma^\prime (\phi^\prime \phi)^{\prime} - \gamma \phi \]

\[= \gamma^\prime \phi^\prime - \gamma \phi = \gamma \phi = 0\]

and

\[(\gamma^\prime \psi - \gamma) \phi = \gamma^\prime \phi \gamma - \gamma \phi = \gamma \phi = 0,\]

Thus,

\[0 \to B \xrightarrow{\psi} X \xrightarrow{\phi} A \to 0\]

\[0 \to B \xrightarrow{\phi} X \xrightarrow{\psi} A \to 0\]

is commutative. With \(X\) and \(X^\prime\) reversed, we get a commutative diagram

\[0 \to B \xrightarrow{\phi} X \xrightarrow{\psi} A \to 0\]

Moreover, the uniqueness in the OMP for pushouts gives \(\psi^\prime = \psi^{-1}\) since a short check shows that \(\psi^\prime \phi\) and \(1_{X^\prime}\) both satisfy the diagram

\[\phi \quad \psi^\prime \]

so \(\psi^\prime \phi = 1_{X^\prime}\). Similarly, \(\psi \psi^\prime = 1_{X}\).