5.5 Convergence

In the previous section we saw that, given a filtration of a complex $C$, there is a spectral sequence $E^p_q$ with $E^0_p = F_p(C_n)/F_{p-1}(C_n)$ and $E^1_{p,q} = H_q(F_p(C))$. Assuming that the filtration is bounded, we will see that this sequence converges to $H_*(C)$. To make this meaningful we need a filtration on $H_*(C)$, we produce a natural filtration arising from that on $C$.

$$F_p(H_*(C)) = \text{im}(H_n(F_p(C)) \to H_n(C)),$$
so

$$F_p(H_*(C)) = \frac{Z_n(F_p(C)) + B_n(C)}{B_n(C)}.$$

**Theorem.** If $C$ is a complex with a bounded filtration then the spectral sequence $E^p_{p,q}$ of 5.4 converges to $H_*(C)$.

**Proof.** Recall the notation from 5.4: $A^p_g = \{ x \in F_p(C_n) | d_k(x) \in F_{p-k}(C_{n-k}) \}$. Since we have a bounded filtration, $F_{p-1}(C_{n-1}) = 0$ for $r > 0$. So, for $r > 0$, $A^p_g$ is constant after a point, and it is $\operatorname{ker}(F_p(C) \to F_p(C_{n-1}))$. We call this value $A^0_p$. Thus, $A^p_g = Z_n(F_p(C))$.

We have the projection map $\pi_p: F_p(C_n) \to E^0_p = F_p(C_n)/F_{p-1}(C_n)$. Restricting the domain to $A^g_p$, we saw that its kernel is $A^g_p \cap F_{p-1}(C_n) = A^g_{p-1}$, so, going to $r > 0$, we see that the kernel of $A^g_p \to E^0_p$ is $A^g_{p-1}$. 


The next thing we need is that \( B_n(C) \cap F_p(C) = U d(A_{p^{r+1},r}) \). The inclusion \( \subseteq \) is clear from the definition of the \( A_{p^r} \). For \( \ni \), take \( d(a) \in F_p(C) \) with \( a \in C \ni t \). We have \( a \in F_{p^{r+1}}(C) = C \ni t \) for large \( r \). Then \( a \in A_{p^{r+1},r} \) by definition. This shows \( \ni \).

Now, to get to the filtration on \( H_n(C) \), we saw that \( A_{p^r} = B_n(F_p(C)) \). So,

\[
F_p(H_n(C)) = \frac{Z_n(F_p(C)) + B_n(C)}{B_n(C)} = \frac{A_{p^r} + B_n(C)}{B_n(C)}
\]

\[
\frac{F_p(H_n(C))}{F_{p-1}(H_n(C))} = \frac{A_{p^r} + B_n(C)}{A_{p^{r-1}} + B_n(C)} = \frac{A_{p^r}}{A_{p^{r-1}} + A_{p^r} \cap B_n(C)}
\]

We next see that \( A_{p^r} \cap B_n(C) = F_p(C) \cap B_n(C) \). The inclusion \( \subseteq \) is clear. For \( \ni \), if \( d(a) \in F_p(C) \), then \( d(d(a)) = 0 \Rightarrow d(a) \in A_{p^r} \) for all \( r \). So, \( d(a) \in A_{p^r} \).

We have a map \( A_{p^r} \xrightarrow{\psi_{p^r}} Z_{p^r} \xrightarrow{\gamma} Z_{p^r} \cap B_{p^r} = E_{p^r} \). Also, since we saw that \( B_{p^r} \subseteq B_{p^{r+1}} \), \( B_{p^r} = U \psi_{p^r}(\nu(A_{p^{r+1},r+1})) \)

\[
= \psi_{p^r}(B_n(C) \cap F_p(C)) \text{ (cancellation)}
\]

So, if \( a \in A_{p^r} \) goes to \( 0 \) in \( E_{p^r} \) (under \( \psi_{p^r} \)), then \( \psi_{p^r}(a) = \psi_{p^r}(a') \) for some \( a' \in A_{p^r} \cap B_n(C) \). So, \( a - a' \in \nu(A_{p^r} \cap B_n(C)) = \nu(A_{p^r} \cap B_n(C)) \), from above. Thus, \( B_n(A_{p^r} \cap E_{p^r}) \subseteq A_{p^{r+1}} + B_n(C) \cap A_{p^r} \). Since the reverse inclusion is clear, we get

\[
\frac{F_p(H_n(C))}{F_{p-1}(H_n(C))} = \frac{A_{p^r}}{A_{p^{r-1}} + A_{p^r} \cap B_n(C)} = B_{p^r}, \text{ as desired}. \]