5.7 Cartan-Eilenberg Resolutions

What we will do in this section is a technical piece of what we need to construct the Grothendieck spectral sequence of the next section.

A Cartan-Eilenberg resolution is somewhat like a projective resolution of a complex. Recall that a projective object in \( 	ext{Ch}(\mathcal{C}) \) is a split exact complex of projectives. Thus, a projective resolution in \( 	ext{Ch}(\mathcal{C}) \) is an exact sequence \( \cdots \rightarrow P_0 \rightarrow A \rightarrow 0 \), where each \( P_i \) is projective in \( 	ext{Ch}(\mathcal{C}) \). This is not what we need. We consider exact sequences \( \cdots \rightarrow P_0 \rightarrow A \rightarrow 0 \) where each \( P_i \) is simply a complex of projectives.

We do need more, which we describe shortly. First, note that we get a double complex from such a sequence. If we write it as

\[
\begin{array}{ccc}
  & P_0 & \rightarrow & P_1 \\
\downarrow & & \downarrow & \\
\cdots & P_0 & \rightarrow & P_{-1} & \rightarrow & 0 \\
\downarrow & & \downarrow & \\
  & A_0 & \rightarrow & A_1 & \rightarrow & 0
\end{array}
\]

then the columns are projective resolutions of the \( A_n \), since a sequence of complexes is exact exactly when, for each \( n \), the corresponding sequence of objects is exact.

Given \( \cdots \rightarrow P_0 \rightarrow A \rightarrow 0 \), for each \( n \) we have induced maps \( B_n(P_n) \rightarrow B_n(A_{n-1}) \), \( Z_n(P_n) \rightarrow Z_n(A_{n-1}) \), and \( H_n(P_n) \rightarrow H_n(A_{n-1}) \). We also have \( B_n(P_0) \rightarrow B_n(A) \), \( Z_n(P_0) \rightarrow Z_n(A) \), and \( H_n(P_0) \rightarrow H_n(A) \). Since these maps are induced by those on the \( P_n \), we have complexes \( \{B_n(P_n)\}, \{Z_n(P_n)\}, \{H_n(P_n)\} \) for each \( r \).
Definition. If $A$ is a complex (with $A_n = 0$ if $n < 0$), then a

Cantam-Eilenberg resolution is a double complex $P$

(with $P_{n0} = 0$ for $p < 0$ or $q < 0$) of projectives, together with

a chain map $P_{*0} \to A$, such that the induced sequences

- $P_{*0} \to A$
- $B_n(P) \to A$
- $Z_n(P) \to Z_n(A)$
- $H_n(P) \to H_n(A)$

are all projective resolutions.

Note, by using double complexes, we need the sign trick
to make the vertical maps be chain maps between rows.
This does not affect well definition of the induced maps
$B_n(P) \to B_n(P_{n-1})$, etc.

Exercise 5.7.1 of [loc:val] shows that the assumption that

$P_{*0} \to A$ and $Z_n(P) \to Z_n(A)$ are projective resolution is

unnecessary.

We show that every complex has a C-E resolution.

Lemma. Every complex in $ch(A)$ has a C-E resolution,
provided that it has enough injectives.

Proof. We make use of the horseshoe lemma. From the

short exact sequence $0 \to B_n(A) \to Z_n(A) \to H_n(A) \to 0$,

we and chosen projective resolutions $B_n(*) \to B_n(A)$

and $H_n(*) \to H_n(A)$, we obtain a projective resolution

$Z_n(*) \to Z_n(A)$ and an exact sequence

$0 \to B_n(*) \to Z_n(*) \to H_n(*) \to 0$ of complexes.
Now, given the short exact sequence
\[ 0 \to \mathbb{Z}_n(A) \to A_n \to B_{n-1}(A) \to 0, \]
there is a projective resolution
\[ P^* \to A_n \text{ and a short exact sequence } 0 \to \mathbb{Z}^n \to P^* \to B_{n-1}^* \to 0. \]

To make \( P = \bigoplus P_{p/q} \) into a double complex, we need horizontal maps. Define \( P_{p/q} \to P_{p/q} \) as the composition \( P_{p/q} \to B_{p/q} \to \mathbb{Z}_{p/q} \to P_{p/q} \). Then this forms a differential, since
\[ P_{p/q} \to B_{p/q} \to \mathbb{Z}_{p/q} \to P_{p/q} \to B_{p/q} \to \mathbb{Z}_{p/q} \to P_{p/q} \to \cdots \]
is clearly 0. We need commutativity of the various squares; the sign trick will then turn \( P \) into a double complex:
\[ P_{p/q} \to B_{p/q} \to \mathbb{Z}_{p/q} \to P_{p/q} \]
\[ P_{p/q-1} \to B_{p/q-1} \to \mathbb{Z}_{p/q-1} \to P_{p/q-1} \]
All three sequences commute since \( P^* \to B^*, B^* \to \mathbb{Z}^*, \) and \( \mathbb{Z}^* \to P^* \) are chain maps. Thus, the large square commutes.

Finally, we need \( B_n(P) = B_n, \mathbb{Z}_n(P) = \mathbb{Z}_n, \) and \( A_n(P) = A_n. \) This is clear since, by construction, the kernel of \( P_{p/q} \to \mathbb{Z}_{p/q} \) is \( \mathbb{Z}_{p/q} \), and the image \( P_{p/q} \to \mathbb{Z}_{p/q} \) is \( B_{p/q} \).
By using injectives, we get a dual notion of a CE complex. If $A$ is a complex, then a right CE resolution of $A$ is a double complex $Q$ of injectives and a chain map $A \to \mathbb{Q}^\bullet$ such that

- $A^p \to Q^p$
- $B^p(A) \to B^p(Q)$
- $Z^p(A) \to Z^p(Q)$
- $H^p(A) \to H^p(Q)$

are all injective resolutions.

The dual of the previous lemma shows that if it has enough injectives, then any cochain complex in $\text{ch}(A)$ has a CE resolution.