Chapter 6: Group Homology and Cohomology

6.1 Definitions and First Properties

In this chapter we study group homology and cohomology and look at two applications of group cohomology: group extensions and finite dimensional division algebras. For these examples we shall see that it is the second cohomology group that we work with, vs the first Ext group when we studied extensions of modules.

The homology and cohomology groups we study here are special cases of Ext and Tor groups, we first study the appropriate ring.

Let $G$ be a group. Then the integral group ring $\mathbb{Z}G$ is a ring defined as follows. As a group it is the free Abelian group on $G$. So, it has a basis indexed by the elements of $G$. For ease of understanding, we view the basis elements as the elements of $G$. So, as a set,

$$\mathbb{Z}G = \{ \sum_{g \in G} n_g g \mid n_g \in \mathbb{Z}, n_g = 0 \text{ a.e.} \}$$

Addition is then “component-wise”: $\sum n_g g + \sum m_g g = \sum (n_g + m_g)g$

For multiplication, we use the multiplication of $G$ and distributivity. More precisely,

$$\sum n_g g \cdot \sum m_g g = \sum_{g} \left( \sum_{h \in G} n_h m_{gh} \right) g$$
The multiplicative identity is $e_0$, and $\mathbb{Z}_b$ is commutative if $b$ is Abelian.

We will consider left $\mathbb{Z}_b$-modules. If $A$ is a left $\mathbb{Z}_b$-module, then for each $g \in \mathbb{Z}_b$ and $a \in A$, $ga \in A$ is defined. The module axioms imply that

$$
g(a+b) = ga + gb$$
$$
g(ha) = (gh)a$$
$$
1 \cdot a = a
$$

In other words, $b$ acts on $A$ as group homomorphisms (automorphisms). Conversely, if there is an operation $b \times A \to A$ on an Abelian group $A$ satisfying these properties, then $A$ is a left $\mathbb{Z}_b$-module via

$$
(\sum_{g} ng) \cdot a = \sum_{g} ng (ga)
$$

We will refer to a $\mathbb{Z}_b$-module as a $b$-module since the $\mathbb{Z}_b$-structure is determined by the $b$-action.

**Example** If $A$ is an Abelian group, then we can make $A$ into a trivial $b$-module via $g \cdot a = a$. We will view $\mathbb{Z}$ as a trivial $\mathbb{Z}_b$-module (for any $b$).

**Example** Let $K/F$ be a Galois extension with $b = Gal(K/F)$. Then $K$ is a $b$-module via $g \cdot a = g(a)$. The field $F$ is a sub $b$-module, note that $F$ is a trivial $b$-module.
Example 2.6 is a $b$-module in the natural way. If $H$ is a subgroup of $b$, then $Zb$ is also an $H$-module. Finally, if $N$ is normal in $b$, then $b/N$ is a $b$-module via $g(nN) = (gn)N$. This is the $b$-module structure induced from the $b/N$-module structure on $b/N$.

Definition If $A$ is a $b$-module, then
\[ H_n(b; A) = Tor^n_b(Z; A), \quad \text{and} \]
\[ H^n(b; A) = Ext^n_b(Z; A). \]

In this definition, as always, we view $Z$ as a trivial $b$-module. So, $H_0(b; A) = \text{Ln}(T)(A)$, where $T = Z \otimes_b -$ and $H^n(b; A) = \text{R}^n(S)(A)$, where $S = \text{Hom}_b(Z, -)$.

To better understand why we consider these groups, and to more easily work with them, we give alternate descriptions by giving alternate descriptions of the functors $\text{Hom}_b(Z, -)$ and $Z \otimes_b -$. Note, we write $\text{Hom}_b(A, B)$ for $\text{Hom}_b(Z; A, B)$. To motivate this, recall that $\text{Hom}(Z, A) \cong A$ via $f \mapsto f(1)$. If $f \in \text{Hom}_b(Z, A)$, then $f(\gamma n) = g f(n)$ for all $n \in Z$. In particular, $f(\gamma n) = g f(n)$. So, $f(a) = f(\gamma 1) a = g \gamma a$.

Definition Let $A$ be a $b$-module.
(i) $A^b = \{ a \in A \mid g \cdot a = a \text{ for all } g \in b \}$
(ii) $A_b = A/\text{I}_b$, $\text{I}_b = \langle \{ g a - a \mid g \in b, a \in A \} \rangle$

= $A/\text{I}_b$, $\text{I}_b$ augmentation ideal.
It is easy to see that $A_b$ is a subgroup of $A$, and that it is a trivial $G$-submodule of $A$. In fact, the definition makes it clear that $A_b$ is the largest trivial submodule of $A$. Dually, $A_b$ is the "largest" homomorphic image of $A$ that is a trivial $G$-module.

we actually have two functors from $G$-modules to $A_b$!

$$S(A) = A_b, \quad T(A) = A_b.$$ 

These act on maps as follows. Given $f : A \to B$, a $G$-module hom,

$$S(f) = f|_{A_b}, \quad T(f) = \left[ \frac{A}{\langle gA - a \rangle} \to \frac{B}{\langle gB - b \rangle} \right]$$

It is easy to see that these are well-defined group homomorphisms, and that $S, T$ are, in fact, functors.

An elementary exercise, which we will not need is that $S$ is right adjoint (resp $T$ is left adjoint) to the trivial module functor $A_b \to G$-mod. Thus, $S$ is left exact and $T$ is right adjoint. However, we can see this in another way.

**Proposition** Let $S$ and $T$ be as above,

1. $S \cong \hom_G(\mathbb{Z}, -)$
2. $T \cong \mathbb{Z} \otimes_{\mathbb{Z}} -$.
proof we prove this for $S$. We define natural transformations $d: S \to F := \text{hom}_B(Z,-)$ and $\beta: F \to S$ by
\[ d_A: A^6 \to \text{hom}_B(Z,A), \quad a \mapsto Ra \quad (n \mapsto na) \]
\[ \beta_A: \text{hom}_B(Z,A) \to A^6 \quad f \mapsto f(1). \]

First, we note that $d_A, \beta_A$ are well-defined group homomorphisms: if $a \in A^6$, then $Ra$ is a $B$-hom since $Ra(\gamma n) = Ra(\gamma) = n a = g(na)$ as $gA = A$. Likewise, $f(1) \in A^6$. If $f \in \text{hom}_B(Z,A)$ then $g \cdot f(1) = f(g1) = f(1)$. It is easy to see $d_A, \beta_A$ preserve $+$. Finally,
\[ \beta_A d_A(a) = \beta_A (Ra) = Ra(1) = a, \]
\[ d_A \beta_A(f) = d_A (f(1)) = R_{\psi}(1) = f, \quad \text{as} \]
\[ R_{\psi}(n) = n f(1) = f(n). \] So, $d_A, \beta_A$ are inverses.

To see $d$ is a natural transformation, let $\psi: A \to B$ be a $B$-hom. We have
\[
\begin{array}{ccc}
A^6 & \xrightarrow{d_A} & \text{hom}_B(Z,A) & \xrightarrow{\beta_A} & A^6 \\
\downarrow \psi A & & \downarrow \psi & & \downarrow \psi \\
B^6 & \xrightarrow{d_B} & \text{hom}_B(Z,B) & \xrightarrow{\beta_B} & B^6
\end{array}
\]

commutes since $\psi \circ \psi(a) = R_{\psi}(a) = \psi \circ Ra = \psi \circ d_A(a)$ since $\psi (Ra(n)) = \psi(na) = n \psi(a) = R_{\psi}(n)$. Also, for $f \in \text{hom}_B(Z,A)$, we have
From this result we have the following alternative description of \( H^n(b,A) \) and \( H^n(b,\mathcal{A}) \):

\[
H^n(b, A) = \text{Ext}^n(\mathcal{A}, A), \quad H^n(b, \mathcal{A}) = \text{Ext}^n(\mathcal{S}, \mathcal{A}).
\]

In particular, if \( 0 \to A \to B \to C \to 0 \) is a s.e.s. of \( b \)-modules, the long exact sequence yields

\[
0 \to A^b \to B^b \to C^b \to H^1(b,A) \to H^1(b,B) \to ... \]

and

\[
... \to H^1(b,B) \to H^1(b,C) \to A^b \to B^b \to C^b \to 0.
\]

Recall how we can compute these groups. For \( H^n(b,A) \), we use projective resolutions. Assuming the result of Ch. 2.7, we can calculate

\[
H^n(b, A) \text{ as the } n \text{th left derived functor of } - \otimes_{b} A \text{ applied to } Z.
\]

So, we form a projective resolution \( P \) of \( Z \) (as \( b \)-modules), then

\[
H^n(b, A) = \text{Hom}(P, A).
\]

We do a similar thing for \( H^n(b, \mathcal{A}) \). By considering the contravariant functor \( \text{Hom}_b(-, A) \), we take a projective resolution \( P \) of \( Z \), and

\[
H^n(b, \mathcal{A}) = \text{Hom}(P, \mathcal{A}).
\]
The benefit of this is that, to calculate $H_n(\mathbb{Z}, A)$ or $H^n(\mathbb{Z}, A)$, we only need to work with projective resolutions of $\mathbb{Z}$; in other words, one projective resolution no matter what is $A$. This will make computations in group cohomology easier than if we had to use resolutions for $A$. The difficulty is that a resolution of $\mathbb{Z}$-modules is not the same thing as a resolution of Abelian groups. So, the fact that $\mathbb{Z}$ had very short resolutions of Abelian groups is not relevant.

We will see how to compute $H_n(\mathbb{Z}, A)$ and $H^n(\mathbb{Z}, A)$ for finite cyclic groups in 6.2 by producing a very nice projective resolution for $\mathbb{Z}$. In 6.3, we will produce a resolution for any finite group that will allow computations.

We finish this section with a simple result about $H_1$. We define the augmentation ideal $I$ to be the kernel of the $6$-homomorphism $\mathbb{Z} 6 \rightarrow \mathbb{Z}$, $g \mapsto 1$. So, $I = \{ \Sigma ng \mid \Sigma ng = 0 \}$.

**Proposition.** $H_1(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}/[6,6]$, the Abelianization of $6$.

**Proof.** The long exact sequence arising from the short exact sequence of $6$-modules

$$0 \rightarrow I \rightarrow \mathbb{Z} 6 \rightarrow \mathbb{Z} \rightarrow 0$$

$$\cdots \rightarrow H_1(\mathbb{Z}, \mathbb{Z}) \rightarrow H_1(\mathbb{Z}, \mathbb{Z}) \rightarrow I^6 \rightarrow (\mathbb{Z} 6)^6 \rightarrow \mathbb{Z} \rightarrow 0.$$
Since \( Z_b \) is free, hence projective, as a \( Z_b \)-module, \( H^1_b(b, Z_b) = \text{Tor}^1_b(b, Z) = 0 \). Thus, we have the exact sequence

\[
0 \to H^1(b, Z) \to I_b \to (Z_b)b
\]

Note that \( I \) has basis \( \{ g^{-1} | g \neq 1 \} \) as a free Abelian group, which is easy to see. So, the definition of \( \text{Tor} \) yields \( A_b = A / IA, \) so,

\[
H^1(b, Z) \cong \ker(I/I^2 \to ZbI). \text{ This kernel is } I/I^2 \text{ so } H^1(b, Z) \cong I/I^2.
\]

We finish the proof by showing \( I/I^2 \cong b^1 = b/\langle b, b \rangle \). To do this, define \( \Theta: b \to I/I^2 \) by \( \Theta(g) = (g-1)+I^2 \).

Then

\[
\Theta(g) = g^{-1} + I^2 = (g^{-1})(h^{-1} + g^{-1}h - 1 + I^2)
\]

\[
= g^{-1}h - 1 + I^2 = g^{-1} + h - 1 + I
\]

Furthermore, \( b^1 = \langle b, b \rangle \) has \( \Theta \) since \( b, b \) is Abelian.

So, we have a hom \( \overline{\Theta}: b^1 \to I/I^2 \) with \( \overline{\Theta} = (g-1) + I^2 \).

To define an inverse as \( I/I^2 \) is free with basis \( g^{-1} + I^2 \), define \( \Psi: I/I^2 \to b^1 \) by \( \Psi(g^{-1} + I^2) = \overline{g} \).

Then

\[
\Psi(\overline{g}) = \Psi(g^{-1} + I^2) = \overline{g},
\]

\[
\overline{\Psi}(g^{-1} + I^2) = \overline{\Psi}(g) = g^{-1} + I^2
\]

So, \( \Psi \) is the inverse of \( \overline{\Theta} \), and so \( \overline{\Theta}: b^1 \to I/I^2 \) is an isomorphism.