6.5 The Bar Resolution

In this section we construct a canonical 6-resolution of \( \mathbb{Z} \) called the bar resolution. By making use of this resolution we can do computations in \( H^n(\mathcal{B},A) \) and \( H_n(\mathcal{B},A) \).

Let \( B_0 \) be the free \( \mathbb{Z} \)-module on one element \([J]\), and for \( n \geq 1 \), let \( B_n \) be the free \( \mathbb{Z} \)-module on \([g_1 \cdots g_n]\) for \( g_i \in \mathbb{G}, i = 1, \ldots, n \). For convenience, we set \([g_1 \cdots g_n] = 0\) if some \( g_i = 1\).

For the differentials, let \( \varepsilon : B_0 \to \mathbb{Z} \) be determined by \( \varepsilon[J] = 1 \). For \( n \geq 1 \), define \( d : B_n \to B_{n-1} \) by

\[
    d([g_1 \cdots g_n]) = g_1 [g_2 \cdots g_n] + \sum_{i=1}^{n-1} (-1)^{i+1} [g_1 \cdots g_i g_{i+1} \cdots g_n]
\]

So,

\[
    n = 1 \quad d([g]) = g [J] - [J] \\
    n = 2 \quad d([fg]) = f [g] - [fg] + [fJ] \\
    n = 3 \quad d([fgi]) = f [g] - [fgi] + [fiJ] - [fi][i] \\
\]

We claim that \( \{B_n, d_n\} \) is a complex. The messy proof we give at the end of the notes for this section.

**Proposition** \( \{B_n, d_n\} \to \mathbb{Z} \) is a free resolution of \( \mathbb{Z} \) as a \( \mathbb{Z} \)-module.

**Proof** We prove that \( \{B_n\} \) is (split) exact by showing \( \delta : B_{-1} \to B_0 \) is null homotopic and checking Exercise 1.4.3 of Weibel.
To do this, define $s_1: \mathbb{Z} \to B_0$ by $s_1(n) = n [1]$ and $s_n: B_n \to B_{n+1}$ (for $n \geq 0$) by $s_n ([g_1, \ldots, g_n]) = [g_1, \ldots, g_n, g_{n+1}]$. Then the $s_n$ are group homomorphisms. We verify that $id = ds + sd$. This will make $\{B_n\}$ a split exact as a complex of Abelian groups, and then give immediately that it is split exact as a complex in $B$-mod.

For $\mathbf{n = 0}$, and writing $d_0 = \varepsilon$,
\[(sd+ds)(g[1]) = s(g[1]) + d([g[1]]) = [1] + g[1] - [1] = g[1],\]
so $sd+ds = \varepsilon B_0$.

For $\mathbf{n \geq 1}$,
\[ds([g_0, g_1, \ldots, g_n]) = d([g_0, g_1, \ldots, g_n]) = [g_0, g_1, \ldots, g_n] - [g_0, g_1, \ldots, g_n] + [g_0, g_1, g_2, \ldots, g_n] + \cdots + (-1)^n [g_0, \ldots, g_{n-1}, g_n] + (-1)^n [g_0, \ldots, g_{n-1}]\]
and
\[sd([g_0, g_1, \ldots, g_n]) = s(g_0, g_1, g_2, \ldots, g_n) - [g_0, g_1, g_2, \ldots, g_n] + \cdots + (-1)^n [g_0, \ldots, g_{n-1}, g_n] + (-1)^n [g_0, \ldots, g_{n-1}]\]
\[= [g_0, g_1, g_2, \ldots, g_n] - [g_0, g_1, g_2, \ldots, g_n] + \cdots + (-1)^n [g_0, \ldots, g_{n-1}, g_n] + (-1)^n [g_0, \ldots, g_{n-1}]\]
so, all terms but one cancel in $(sd+ds)(g_0, g_1, \ldots, g_n)$, giving just $g_0 [g_1, \ldots, g_n]$.

Thus, $id = ds + sd$, as desired.

Alternatively, we could use the "nonnormalized" bar resolution, where $B_n$ is the free $\mathbb{Z}B$-module on $\{g_1, \ldots, g_n\}$ if $n = 0$, and $B_n$ is free on $\{1\}$, and where
\[d_0([g_0, g_1, \ldots, g_n]) = [1] (g_1, g_2, \ldots, g_n) - (g_0 g_1, g_2, \ldots, g_n) + \cdots + (-1)^n (g_0, g_1, g_2, \ldots, g_{n-1}, g_n) + (-1)^n (g_0, g_1, g_2, \ldots, g_{n-1}).\]
To calculate \( H^n(b, A) \) we take the homology of \( \{ \text{hom}_b(B^n, A) \} \) or \( \{ \text{hom}_b(B_n, A) \} \). By the UMP of free modules, we can identify \( \text{hom}_b(B^n, A) = \text{Maps}(B^n, A) \). The induced differentials are

\[
d(f)(g_0, \ldots, g_n) = d(f(g_0, \ldots, g_n)) = g_0 f(g_0, \ldots, g_n) - f(g_0g_1g_2, \ldots, g_n) + \cdots + (-1)^n f(g_0 \cdots g_{n-1}).
\]

We call elements of \( \text{Maps}(B^n, A) \) \( n \)-cochains, elements of \( \text{ker} \, d^n \) \( n \)-co-cycles, and those in \( \text{im} \, d^{n-1} \) \( n \)-coboundaries. We write \( Z^n(b, A) = \text{ker} \, d^n \) and \( B^n(b, A) = \text{im} \, d^{n-1} \), so

\[
H^n(b, A) = Z^n(b, A) / B^n(b, A).
\]

If we use \( \{ B_n \} \) instead of \( \{ B^n \} \), then \( \text{hom}_b(B_n, A) \) consists of "normalized" cochains, maps \( f : b^n \rightarrow A \) with \( f(g_0, \ldots, g_n) = 0 \) if some \( g_i = 1 \). We then have normalized cocycles and normalized coboundaries.

A 1-cocycle is a function \( f : b \rightarrow A \) with

\[
f(g h) = g f(h) + f(g), \quad \text{as } d(f) = 0.
\]

Such an element is a derivation of \( b \) (into \( A \)); by making \( A \) into a trivial right \( b \)-module, we have

\[
f(g h) = g f(h) + f(g) h, \quad \text{which looks like the product rule.}
\]

A 1-coboundary is a function \( f = d(g \ast \phi) \), so where \( f(g) = g \alpha - \alpha \) for some \( \alpha \). Here, \( \ast : b^0 = \phi ! \rightarrow A \) picks a point at \( A \). Thos., \( f(g) = g \alpha - \alpha g \), a "principal" derivation.

Thos., \( H^1(b, A) = \text{Der}(b, A) / \text{PDer}(b, A) \), which is Theorem 6.4.5 of Weibel.

Also note that if \( A \) is a trivial left \( b \)-module, then \( H^1(b, A) = \text{Hom}(b, A) \).
A 2-cocycle is a function $f: b \times b \to A$ with
$$lf(g,h,h) = 0 \text{ or } g + (h,h) - f(g,h) + f(g,h,h) - f(g,h,h,h) = 0.$$ In other words,
$$f(g,h) + f(gh,b) = g f(h,b) + f(g,h,h),$$
we will see this formula arising in two natural ways in $b \times b$.

A 2-coboundary $f = d(\sigma)$ is a function given by
$$f(g,h) = \sigma(gh) - \sigma(h) + \sigma(g)$$ for some function $\sigma: b \to A$.
If we write $\sigma(h) = 0$, then $f$ is given by
$$f(g,h) = g + g\alpha_h - g\alpha_h.$$ We now prove a result about group cohomology (and homology) using the bar resolution and dimension shifting.

**Proposition** Let $b$ be a finite group of order $r$. Then $H^n(b, A)$ is annihilated by $r$ for any $n \geq 1$.

**Proof** Let $f$ be a 1-cocycle $b \to A$. We show $rf$ is a 1-coboundary. We have $f(gh) = g f(h) + f(g)$ for all $g, h \in b$. So, $f(g) = f(gh) - g f(h)$. Summing over all $h \in b$ gives $rf(g) = \sum f(gh) - g f(h)$. If $a = \sum f(h)$, then $a = \sum f(gh)$, so $f(g) = a - g a = g(-\alpha_a) + (-a)$. Thus, $f$ is a 1-coboundary. Thus, $H^1(b, A) = 0$ for any $A$.

Now, if $\mathcal{P}_n$ is a projective resolution of $A$, and if $\text{ker}(P_n \to P_{n-1})$ we have seen that $H^n(b, \mathcal{P}_n) = H^n(b, \mathcal{P}_{n-1})$. So, $H^n(b, A) = 0$ for all $n \geq 1$. \qed
We now prove that $L_n^1, L_n^2$ is a complex. We do this by relating it to perhaps a more familiar object. Let $P_n$ be the free Abelian group on $S_n$ made into a $b$-module by $g_1 g_2 \cdots g_n = (g_1 g_2 \cdots g_n).$ It is not hard to check that $P_n$ is a free $b$-module with basis $\{1, g_1, \ldots, g_n\} \subseteq S_n.$ Thus, $P_n \cong b_n^1.$ However, we see an isomorphism that is not the obvious one. Define $\phi: P_n \rightarrow b_n^1$ by

$$\phi(g_0, \ldots, g_n) = g_0 (g_0^1, g_0^2, \ldots, g_n^{-1} g_n).$$

Its inverse satisfies $\varphi(g_0, \ldots, g_n) = (1, g_1, g_1 g_2, \ldots, g_1 \cdots g_n).$ To see this, write $(g_0 g_1, \ldots, g_0 \cdots g_n) = (h_1, \ldots, h_n).$ Then $h_1 h_2 = g_0 g_1, \quad h_1 h_3 = g_0 g_1 g_2, \quad \ldots, \quad h_n = g_0 g_1 \cdots g_n.$ Since $\sigma(1, g_0 g_1, \ldots, g_0 \cdots g_n) = (h_1, \ldots, h_n), we see that\]

$$\varphi^{-1}(h_1, \ldots, h_n) = (1, h_1, h_1 h_2, \ldots, h_1 \cdots h_n).$$

Now, consider $L_1: P_n \rightarrow P_{n-1}$ given by $L_1(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n).$ We show $L_1 L_1 = 0.$

$$L_1(L_1(g_0, \ldots, g_n)) = L_1 \left( \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n) \right)$$

$$= \sum_{i=0}^{n} \sum_{j<i} (-1)^{i+j} (g_0, \ldots, \hat{g}_i, \ldots, \hat{g}_j, \ldots, g_n) + \sum_{i=0}^{n} \sum_{j>i} (-1)^{i+j} (g_0, \ldots, \hat{g}_i, \ldots, g_j, \ldots, g_n).$$

The reason for the $L_1 L_1$ expression is that the removal of $g_j$ from $(g_0, \ldots, \hat{g}_i, \ldots, g_n)$ is removing the $j-1$ term.

Looking at the first sum, the pairs $(i,j)$ in the sum are

$$\begin{align*}
(1,0), \\
(1,1), \\
(2,1), \\
(2,2), \\
\vdots \\
(n,n-1).
\end{align*}$$

So, reversing $i,j$, we see that

$$\sum_{i=0}^{n} \sum_{j<i} (-1)^{i+j} (g_0, g_1, \ldots, \hat{g}_i, \ldots, g_n) = \sum_{i=0}^{n} \sum_{j<i} (-1)^{i+j} (g_0, g_1, \ldots, \hat{g}_i, \ldots, g_n).$$
So, \( d \left( d(g_0, \ldots, g_n) \right) = 0 \).

Now, consider the diagram

\[
\begin{array}{c}
\gamma \uparrow \\
\overset{\sigma}{\downarrow} \\
B_0 \overset{d}{\longrightarrow} B_{n-1}
\end{array}
\]

The composition yields differentials on \( \mathbb{B}_n^{\leq 3} \). We have

\[
(g_0, \ldots, g_n) \rightarrow (1, g_0, g_1, \ldots, g_n) \\
\rightarrow (g_0, g_1, g_2, \ldots, g_n) + (g_0, g_1, g_2, \ldots, g_n) + \ldots + (-1)^n (g_0, g_1, g_2, \ldots, g_n)
\]

\[
\rightarrow g_1 (g_2, \ldots, g_n) - (g_1, g_2, \ldots, g_n) + \cdots + (-1)^n (g_1, g_2, \ldots, g_n)
\]

since

\[
\tau \left( 1, g_0, \ldots, g_n \right) = (g_0, g_1, g_2, \ldots, g_n)
\]

So, \( d \sigma \) is our original \( d : \mathbb{B}_n^{\leq 3} \rightarrow \mathbb{B}_n^{\leq 3} \). Thus, an easy diagram chase shows that \( \mathbb{B}_n^{\leq 3}, d \) is a complex.