Ext Groups and Ext Functors

In this note we discuss the Hom and Ext functors and their connection with extensions of Abelian groups. The theory we develop has an analogue in the category of R-modules for any ring R; however, we will restrict to $R = \mathbb{Z}$ by considering only Abelian groups. A nice reference for this material is Chapter 11 of *Introduction to the Theory of Groups*, (3rd ed.) by Rotman.

1 The Hom Functors

If A and B are Abelian groups, then $\operatorname{Hom}(A, B)$ is also an Abelian group under pointwise addition of functions. In this section we will see how Hom gives rise to classes of functors. Let $\mathcal{A}$ denote the category of Abelian groups. A *covariant functor* $T : \mathcal{A} \to \mathcal{A}$ associates to every Abelian group A an Abelian group $T(A)$, and for every homomorphism $f : A \to B$ a homomorphism $T(f) : T(A) \to T(B)$, such that the following properties hold:

1. $T(\mathrm{id}_A) = \mathrm{id}_{T(A)}$ for all $A \in \mathcal{A}$;
2. If $f : A \to B$ and $g : B \to C$, then $T(g \circ f) = T(g) \circ T(f)$;
3. If $f, g : A \to B$, then $T(f + g) = T(f) + T(g)$.

If, instead, (2) is replaced by the equation $T(g \circ f) = T(f) \circ T(g)$, we say that T is a *contravariant functor*. Moreover, the definition of a functor usually requires only the first two conditions. Adding (3) usually goes under the name additive functor.

Example 1.1. Let M be an Abelian group, and let $T = \operatorname{Hom}(M, -)$. In other words, T is the functor for which $T(A) = \operatorname{Hom}(M, A)$. To define the action of T on maps, let $f : A \to B$ be a homomorphism. We want a map $T(f) : \operatorname{Hom}(M, A) \to \operatorname{Hom}(M, B)$. By drawing a diagram, we see that $\alpha \mapsto f \circ \alpha$ is such a homomorphism. We define $T(f)$ by $T(f)(\alpha) = f \circ \alpha$. It is then easy to see T is a covariant functor. On the other hand, if we define $S = \operatorname{Hom}(-, M)$, then we have a contravariant functor, which acts on maps as follows: If $f : A \to B$, then $S(f) : \operatorname{Hom}(B, M) \to \operatorname{Hom}(A, M)$ is given by $S(f)(\alpha) = \alpha \circ f$. Note the position of f in this formula. Since S reverses the order of maps, it is contravariant.

To understand better the Hom functors, we need the following property. A covariant functor T is said to be *left exact* if for every exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the induced sequence $0 \to T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$ is exact. A contravariant functor S is left exact if for every such sequence, the induced sequence $S(C) \xrightarrow{S(g)} S(B) \xrightarrow{S(f)} S(A) \to 0$ is exact. Right exactness is defined in a similar way. If $T = \mathrm{Hom}(M, -)$ (resp. $S = \mathrm{Hom}(-, M)$), we will write f_* (resp. f^*) for $T(f)$ (resp. $S(f)$) if $f : A \to B$ is a homomorphism. While we do not need it here, the tensor product functor is right exact.

Proposition 1.2. *Let M be an Abelian group. Then $\mathrm{Hom}(M, -)$ is a left exact covariant functor and $\mathrm{Hom}(-, M)$ is a left exact contravariant functor.*

Proof. We prove the first statement; the second is analogous. Suppose that $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact. We need to show that $0 \to \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C)$ is exact. First, we show that f_* is injective. Let $\alpha \in \mathrm{Hom}(M, A)$ with $f_*(\alpha) = 0$. Then $f \circ \alpha = 0$. Thus, $\alpha(M) \subseteq \ker(f) = 0$. This yields $\alpha = 0$. Next, if $\alpha \in \mathrm{Hom}(M, A)$, we have $g_*(f_*(\alpha)) = g \circ (f \circ \alpha) = (g \circ f) \circ \alpha = 0$. This proves $\mathrm{im}(f_*) \subseteq \ker(g_*)$. Finally, we must show that $\ker(g_*) \subseteq \mathrm{im}(f_*)$. To do this, suppose that $\beta : M \to B$ is an element of $\ker(g_*)$. Then $g \circ \beta = 0$. Thus, $\beta(M) \subseteq \ker(g) = \mathrm{im}(f)$. Therefore, for each $m \in M$ there is an $a \in A$ with $\beta(m) = f(a)$. The element a is unique since f is injective. Thus, we may define a function $\alpha : M \to A$ by $\alpha(m) = a$. In other words, $\alpha(m) = f^{-1}(\beta(m))$; we interpret f^{-1} as a homomorphism from $\mathrm{im}(f)$ to A. The map α is the composition of two homomorphisms, so it is an element of $\mathrm{Hom}(M, A)$. Moreover, $f_*(\alpha)(m) = f(\alpha(m)) = \beta(m)$. Therefore, $f_*(\alpha) = \beta$; thus, $\beta \in \mathrm{im}(f_*)$. This finishes the proof. $\square$

Example 1.3. It is not true that, in general, $\mathrm{Hom}(M, -)$ is right exact. To give an example, let $M = \mathbb{Z}_p$, and consider the exact sequence $0 \to p\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_p \to 0$. Applying $\mathrm{Hom}(M, -)$ yields the sequence $0 \to \mathrm{Hom}(\mathbb{Z}_p, p\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_p)$. However, the first two terms are 0; there are no nontrivial homomorphisms from the finite group $\mathbb{Z}_p$ into the group $\mathbb{Z} \cong p\mathbb{Z}$. Moreover, $\mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a nontrivial group. The map $\mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}_p, \mathbb{Z}_p)$ is thus not surjective.

In some sense, homological algebra arose from the failure of Hom (and other functors) to fail to be both right and left exact. We will define new functors, the Ext functors, which explain this failure. Before we do this, we show for which groups the corresponding Hom functors are exact. To aid in the understanding of the following proposition, we recall some terms from a graduate algebra course. An Abelian group P is said to be *projective* if for every diagram

$$\begin{tikzcd} & P \arrow[d, "\beta"] & \\ B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

there is a homomorphism $\alpha : P \to B$ with $g \circ \alpha = \beta$. In other words, the diagram can be completed to the following commutative diagram

$$\begin{tikzcd}
 & P \arrow[dl, dotted, "\alpha"'] \arrow[d, "\beta"] & \\
B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

Similarly, an Abelian group I is *injective* if for every diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \\
 & I &
\end{tikzcd}$$

there is a map $\beta : B \to I$ with $\beta \circ f = \alpha$,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[dl, dotted, "\beta"] \\
 & I &
\end{tikzcd}$$

yielding the commutative diagram above. We will show that, in the category of Abelian groups, projectives are the same thing as free Abelian groups. To help us do this, we need the following lemma, which is actually more general than the statement that projectives are free. In the proof of this lemma, we use the fact that $\mathbb{Z}$ is projective, and so that a short exact sequence of the form $0 \to K \to F \to \mathbb{Z} \to 0$ is split exact.

Lemma 1.4. *A subgroup of a free Abelian group is free Abelian.*

Proof. Let P be a subgroup of a free group F. We give the proof in the case that the basis of F is countable; let $\{e_i : i \in \mathbb{N}\}$ be a basis for F. Then $F = \bigoplus_{i=1}^{\infty} \mathbb{Z}e_i$. Set $F_n = \bigoplus_{i=1}^{n} \mathbb{Z}e_i$. Then $F_{n+1}/F_n \cong \mathbb{Z}$ for each n. Write $P_j = P \cap F_j$. Then P_{n+1}/P_n is isomorphic to a subgroup of $F_{n+1}/F_n \cong \mathbb{Z}e_{n+1}$. Thus, either $P_{n+1}/P_n = 0$ or $P_{n+1}/P_n \cong \mathbb{Z}$. However, in the latter case, from the exact sequence $0 \to P_n \to P_{n+1} \to \mathbb{Z} \to 0$, we see, since $\mathbb{Z}$ is free, that $P_{n+1} \cong P_j \oplus \mathbb{Z}$. Let A_n be a summand of P_{n+1} such that $P_{n+1} = P_n \oplus A_n$. By induction, it follows that $P_n \cong \bigoplus_{i=1}^{n} A_i$. Since P is the union of all the P_n, we conclude that $P \cong \bigoplus_{i=1}^{\infty} A_i$. From this we see that P is free, since it is the direct sum of the nonzero A_i, all of which are isomorphic to $\mathbb{Z}$. The argument for an arbitrary sized basis $\{e_i : i \in I\}$ goes through much the same as that above once we well order the index set I and use transfinite induction. $\square$

The analogue of this result in the category of R-modules is false: submodules of free R-modules are usually not free. For an easy example, if $R = \mathbb{Z}_6$, then $2R$ is a submodule of order 3. A direct sum of copies of R has order at least 6, so $2R$ is not free. We will see how this fact about Abelian groups has applications to the Ext functors we will define below.

Lemma 1.5. *An Abelian group P is projective if and only if P is free Abelian. Also, an Abelian group I is injective if and only if I is divisible.*

Proof. Recall that a group A is divisible if for every $a \in A$ and $n \in \mathbb{N}$, there is $b \in A$ with $nb = a$. To prove the second statement first, suppose that I is injective. Let $a \in I$ and $n \in \mathbb{N}$. From the sequence $0 \to n\mathbb{Z} \to \mathbb{Z}$ and the homomorphism $\alpha : n\mathbb{Z} \to I$ given by $\alpha(nm) = ma$, there is a homomorphism $\beta : \mathbb{Z} \to I$ with $\beta|_{n\mathbb{Z}} = \alpha$. If $b = \alpha(1)$, then $nb = \alpha(n) = a$. Thus, I is divisible. Conversely, suppose that I is divisible. Let $0 \to A \xrightarrow{f} B$ be exact and let $\alpha : A \to M$ be a homomorphism. We use a Zorn's lemma argument to extend α to B. We view $A \subseteq B$ by identifying A with its isomorphic image $f(A)$. Let $\mathcal{S}$ be the set of all pairs (C, γ) with C a subgroup of B satisfying $A \subseteq C$ and $\gamma : C \to I$ with $\gamma|_A = \alpha$. Ordering $\mathcal{S}$ via $(C, \gamma) \leq (C', \gamma')$ if $C \subseteq C'$ and $\gamma'|_C = \gamma$. A short argument shows that we can apply Zorn's lemma to obtain a maximal element (B_0, β). We claim that $B_0 = B$. If not, there is a $b \in B - B_0$. Let $B_1 = B_0 + \mathbb{Z}b$, the group generated by B_0 and b. If $B_0 \cap \mathbb{Z}b = 0$, we may extend β by defining $\beta'(x + nb) = \beta(x)$ for every n. If $B_0 \cap \mathbb{Z}b \neq 0$, then, as a subgroup of the cyclic group $\mathbb{Z}b$, there is an m with $B_0 \cap \mathbb{Z}b = \mathbb{Z}mb$. Since I is divisible and $mb \in B_0$, there is an $r \in I$ with $mr = \beta(mb)$. We then define β' on $B_0 + \mathbb{Z}b$ by $\beta'(x + nb) = \beta(x) + nr$. A short argument shows that this extension is well-defined and is a homomorphism. Maximality gives a contradiction; thus, $B_0 = B$. Therefore, we have a map $\beta : B \to I$ with $\beta \circ f = \alpha$, as desired.

For the first statement, first suppose that P is free Abelian, and let $B \xrightarrow{\pi} C \to 0$ be exact, and $f : P \to C$ be a homomorphism. Let X be a basis of P as a free Abelian group. For each $x \in X$, choose $b \in B$ with $\pi(b) = f(x)$. Define $g : P \to B$ to be the unique homomorphism arising from the function $X \to B$ given by $x \mapsto b$. Then $\pi \circ g = f$. Conversely, suppose that P is projective. It is then a summand of a free group. By the previous lemma, P is free Abelian. □

Proposition 1.6. *The functor* $\mathrm{Hom}(M, -)$ *is exact if and only if M is a free Abelian group. The functor* $\mathrm{Hom}(-, M)$ *is exact if and only if M is a divisible group.*

Proof. By the lemma, it is enough to prove that $\mathrm{Hom}(M, -)$ is exact if and only if M is projective, and $\mathrm{Hom}(-, M)$ is exact if and only if M is injective. The arguments for these two statements are quite similar, so we prove only the first. Suppose that M is projective, and let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. We know that $0 \to \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C)$ is exact. It suffices to prove that g_* is surjective. Take $\beta \in \mathrm{Hom}(M, C)$. We then have the diagram

$$\begin{tikzcd} & M \arrow[d, "\beta"] & \\ B \arrow[r, "g"] & C \arrow[r] & 0. \end{tikzcd}$$

Thus, since M is projective, there is $\alpha \in \mathrm{Hom}(M, B)$ with $g \circ \alpha = \beta$. In other words, $g_*(\alpha) = \beta$. Thus, g_* is surjective. Conversely, suppose that $\mathrm{Hom}(M, -)$ is exact. To

show M is projective, suppose that there is a surjection $g : B \to C$ and a homomorphism $\beta : M \to C$. We have an exact sequence $0 \to \ker(g) \to B \xrightarrow{g} C \to 0$. Applying $\mathrm{Hom}(M, -)$ to this sequence and using exactness yields the exact sequence $0 \to \mathrm{Hom}(M, \ker(g)) \to \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \to 0$. Since g_* is surjective, there is $\alpha \in \mathrm{Hom}(M, B)$ with $g_*(\alpha) = \beta$. This says $g \circ \alpha = \beta$. Thus, M is projective. $\square$

We will produce Ext functors which will have the property that given an exact sequence $0 \to A \to B \to C \to 0$, we obtain an exact sequence, for every group M, of the form

$$\begin{aligned} 0 \to \mathrm{Hom}(M, A) &\to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to \\ \mathrm{Ext}(M, A) &\to \mathrm{Ext}(M, B) \to \mathrm{Ext}(M, C) \to 0, \end{aligned}$$

along with a related sequence for the functor $\mathrm{Hom}(-, M)$. Thus, the Ext functors in some sense repair the failure of exactness.

2 Ext(C,A)

When we discussed group extensions of an Abelian group A by a group G, we saw that if G acts trivially on A and if we have a symmetric cocycle, the corresponding group extension is an Abelian group. In other words, the extension we obtained is an object in the category $\mathcal{A}$. To keep ourselves in this category, if A and C are Abelian groups with C acting trivially on A, we say that $z : C \times C \to A$ is a *symmetric cocycle* if $z(0, c) = z(c, 0) = 0$ for all $c \in C$, if $z(c, d) = z(d, c)$ for all $c, d \in C$, and if z satisfies the cocycle condition

$$z(c, d) + z(c + d, e) = z(d, e) + z(c, d + e).$$

We will denote by $Z(C, A)$ the group of all symmetric cocycles. A *symmetric coboundary* is a symmetric cocycle of the form $b(c, d) = l(c) + l(d) - l(c + d)$ for some function $l : C \to A$. We denote by $B(C, A)$ the group of all symmetric coboundaries. We then define

$$\mathrm{Ext}(C, A) = Z(C, A)/B(C, A).$$

In other words, $\mathrm{Ext}(C, A) = H^2(C, A)_{\mathrm{sym}}$. From what we did for group extensions, the elements of $\mathrm{Ext}(C, A)$ are in 1-1 correspondence with equivalence classes of Abelian group extensions of C by A. We recall this correspondence. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an extension of A by C. Choose a function $l : C \to B$ with $g \circ l = \mathrm{id}_C$. Then the cocycle z associated to l is the function given by

$$z(c_1, c_2) = f^{-1}((l(c_1) + l(c_2) - l(c_1 + c_2)) .$$

To understand this formula, we see that $l(c_1) + l(c_2) - l(c_1 + c_2) \in \ker(g) = \mathrm{im}(f)$. Since f is injective, there is a unique element a such that $f(a) = l(c_1) + l(c_2) - l(c_1 + c_2)$. Then $z(c_1, c_2) = a$. The group $\mathrm{Ext}(C, A) = 0$ if and only if the only extension of A by C is the

direct product $A \times C$. Alternatively, $\mathrm{Ext}(C, A) = 0$ if and only if every exact sequence of the form $0 \to A \to B \to C \to 0$ is split exact. Therefore, we see that $\mathrm{Ext}(A, D) = 0$ if D is divisible, and $\mathrm{Ext}(F, A) = 0$ if F is free Abelian. Since we know that $\mathrm{Hom}(-, D)$ is exact if D is divisible and $\mathrm{Hom}(F, -)$ is exact if F is free, we do not need to repair exactness of these functors via Ext functors; this rationale argues that, however we were to define Ext, we should have these facts about Ext.

3 Pullbacks and Pushouts

We know, by the definition of addition of cocycles, that $\mathrm{Ext}(C, A)$ is an Abelian group in 1-1 correspondence with the set of extensions of A by C. Thus, we can translate this bijection to give a definition of addition of extensions. We will describe this addition; to do so, we first discuss pullbacks and pushouts.

Suppose that we have the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ C & \end{tikzcd}$$

of Abelian groups. The *pushout* of this diagram is an Abelian group P, together with maps $\alpha : C \to P$ and $\beta : B \to P$, such that $\beta \circ f = \alpha \circ g$, and such that the following mapping property holds: If Q is an Abelian group together with homomorphisms $\alpha' : C \to Q$ and $\beta' : B \to Q$ such that $\beta' \circ f = \alpha' \circ g$, then there is a unique homomorphism $h : P \to Q$ such that $h \circ \alpha = \alpha'$ and $h \circ \beta = \beta'$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d, "\beta"] \arrow[ddr, bend left, "\beta'"] & \\ C \arrow[r, "\alpha"] \arrow[drr, bend right, "\alpha'"] & P \arrow[dr, "h"] & \\ & & Q \end{tikzcd}$$

The dual of this notion is called a pullback. If we have the following diagram

$$\begin{tikzcd} & B \arrow[d, "f"] \\ C \arrow[r, "g"] & A \end{tikzcd}$$

then the *pullback* is an Abelian group P, together with maps $\alpha : P \to B$ and $\beta : P \to C$ satisfying $g \circ \beta = f \circ \alpha$, such that for every Abelian group Q with maps $\alpha' : Q \to B$ and $\beta' : Q \to C$ satisfying $g \circ \beta' = f \circ \alpha'$, there is a unique homomorphism $h : Q \to P$ with

$\alpha \circ h = \alpha'$ and $\beta \circ h = \beta'$.

$$\begin{tikzcd}
Q \arrow[dr, "h"] \arrow[drr, bend left, "\alpha'"] \arrow[ddr, bend right, "\beta'"'] & & \\
& P \arrow[r, "\alpha"] \arrow[d, "\beta"] & B \arrow[d, "f"] \\
& C \arrow[r, "g"] & A
\end{tikzcd}$$

We show that pushouts and pullbacks exist. To construct pushouts, suppose that we have maps $f : A \to B$ and $g : A \to C$. We define $P = (B \oplus C)/S$, where $S = \{(f(a), -g(a)) : a \in A\}$. It is clear that S is a subgroup of $B \oplus C$. We have maps $\beta : B \to P$ and $\alpha : C \to P$ given by $\beta(b) = (b, 0) + S$ and $\alpha(c) = (0, c) + S$. Furthermore, the definition of S is exactly what we need to see that $\beta \circ f = \alpha \circ g$, since

$$\beta(f(a)) = (f(a), 0) + S = (0, g(a)) + S = \alpha(g(a))$$

for all $a \in A$. To show the mapping property, suppose that we have maps $\beta' : B \to Q$ and $\alpha' : C \to Q$ with $\beta' \circ f = \alpha' \circ g$. By the mapping property for direct sum, we have a map $B \oplus C \to Q$, given by $(b, c) \mapsto \beta'(b) + \alpha'(c)$. The kernel of this map contains S since $\beta' \circ f = \alpha' \circ g$; therefore, there is an induced map $h : P \to Q$, given by $h((b, c) + S) = \beta'(b) + \alpha'(c)$. From this formula it follows that $h \circ \beta = \beta'$ and $h \circ \alpha = \alpha'$. Finally, if $k : P \to Q$ is another map with $k \circ \beta = \beta'$ and $k \circ \alpha = \alpha'$, then $\beta'(b) = k(\beta(b)) = k((b, 0) + S)$ and $\alpha'(c) = k(\alpha(c)) = k((0, c) + S)$. Combining these two equations, we get

$$k((b, c) + S) = k((b, 0) + S) + k((0, c) + S) = \beta'(b) + \alpha'(c).$$

Thus, $k = h$. Therefore, P is the pushout of the diagram.

To construct pullbacks, we use similar ideas. Instead of constructing a quotient of a direct sum, we use a subgroup of a direct product. Given $f : B \to A$ and $g : C \to A$, we set $P = \{(b, c) \in B \oplus C : f(b) = g(c)\}$. The canonical projection maps $B \oplus C \to B$ and $B \oplus C \to C$ restrict to P to yield maps $\alpha : P \to B$ and $\beta : P \to C$. From the definition of P it is clear that $f \circ \alpha = g \circ \beta$. Suppose that there is a group Q and maps $\alpha' : Q \to B$ and $\beta' : Q \to C$ with $f \circ \alpha' = g \circ \beta'$. The mapping property for direct product gives a map $Q \to B \oplus C$, defined by $q \mapsto (\alpha'(q), \beta'(q))$. The equation $f \circ \alpha' = g \circ \beta'$ shows that the image of Q is contained in P. Thus, we view this as a map $h : Q \to P$, and we have, by the definition of h, that $\alpha \circ h = \alpha'$ and $\beta \circ h = \beta'$. Moreover, an argument similar to that for pushouts, we see that h is unique. Therefore, P is the pullback of the diagram.

A *morphism* (α, β, γ) of extensions $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and $0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ is a triple of homomorphisms such that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

is commutative. An equivalence of two extensions $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and $0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C \to 0$ of A by C is then a map $(\mathrm{id}_A, \beta, \mathrm{id}_C)$. It is a consequence of the snake lemma below that β is automatically an isomorphism.

Lemma 3.1 (Snake Lemma). *Suppose that*

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' & \end{tikzcd}$$

is a commutative diagram. Then there is an exact sequence

$$\ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \to \mathrm{coker}(\alpha) \to \mathrm{coker}(\beta) \to \mathrm{coker}(\gamma).$$

In particular, if α and γ are injective (resp. surjective), then β is injective (resp. surjective).

Proof. The proof of this result is just a long exercise in playing with commutative diagrams. For a full proof, see a book on homological algebra. We will only prove the consequence stated in the final sentence. First, suppose that α and γ are injective, and suppose that $\beta(b) = 0$. Then $0 = g'\beta(b) = \gamma g(b)$. Since γ is injective, $g(b) = 0$, so $b = f(a)$ for some $a \in A$. Then $0 = \beta(b) = \beta f(a) = f'\alpha(a)$. Since both α and f' are injective, $a = 0$, so $b = f(a) = 0$. Next, suppose that α and γ are surjective, and take $b' \in B'$. Then $g'(b') = \gamma(c)$ for some $c \in C$. Write $c = g(b)$ for some $b \in B$. Then $g'(b') = \gamma(c) = \gamma g(b) = g'\beta(b)$. Thus, $b' - \beta(b) = f'(a')$ for some $a' \in A'$. Writing $a' = \alpha(a)$ for some $a \in A$, we have

$$\begin{aligned} b' &= \beta(b) + f'(a') = \beta(b) + f'\alpha(a) = \beta(b) + \beta f(a) \\ &= \beta(b + f(a)). \end{aligned}$$

Thus, β is surjective. $\square$

The snake lemma is at the heart of several facts in homological algebra, most notably the construction of the connecting homomorphisms of homology and cohomology groups. However, when we discuss the Ext groups below, we give ad-hoc constructions of the connecting homomorphisms since we would have to discuss more homological algebra in order to make use of the snake lemma.

The following two lemmas are the keys to constructing the Ext functors. To motivate their statements, if A is an Abelian group, then we will prove that $\mathrm{Ext}(-, A)$ is a contravariant functor. This requires us to construct, for each homomorphism $\gamma : C' \to C$, a homomorphism $\gamma^* : \mathrm{Ext}(C, A) \to \mathrm{Ext}(C', A)$. Similarly, if $\alpha : A \to A'$ is a map and C a group, to prove that $\mathrm{Ext}(C, -)$ is a covariant functor, we need to produce a map $\alpha_* : \mathrm{Ext}(C, A) \to \mathrm{Ext}(C, A')$. In terms of extensions, the first statement requires us to produce, for an exact sequence

$\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, an exact sequence $\mathcal{E}' : 0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f'"] \arrow[d, "\mathrm{id}"] & B' \arrow[r, "g'"] \arrow[d, "\beta"] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative. In other words, we must produce a map $(\mathrm{id}_A, \beta, \gamma) : \mathcal{E}' \to \mathcal{E}$. The second situation requires us to produce a map $(\alpha, \beta, \mathrm{id}_C) : \mathcal{E} \to \mathcal{E}'$. We will produce the required extensions by means of pullbacks and pushouts.

Lemma 3.2. *Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an extension and let $\alpha : A \to A'$ be a homomorphism. Then there is an extension $\mathcal{E}' : 0 \to A' \xrightarrow{f'} B' \xrightarrow{g'} C \to 0$, unique up to equivalence, and a map $(\alpha, \beta, \mathrm{id}_C) : \mathcal{E}' \to \mathcal{E}$ of extensions. Furthermore, if z is a cocycle representing $\mathcal{E}$, then $\alpha \circ z$ is a cocycle representing $\mathcal{E}'$.*

Proof. Let B' be the pushout of the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \\
A' &
\end{tikzcd}$$

We then have maps $\beta : B \to B'$ and $f' : A' \to B'$. Furthermore, from the map $g : B \to C$ and the zero map $A' \to C$, the UMP for pushouts yields a unique map $g' : B' \to C$, yielding the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
 & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C &
\end{tikzcd}$$

Because the right square is commutative, the map g' is surjective. The composition $g'f' = 0$ by the definition of pushout. Finally, we must show that $\ker(g') = \mathrm{im}(f')$ and that f' is injective. We do both by using the explicit description of $B' = (A' \oplus B)/S$, where $S = \{(\alpha(a), -f(a)) : a \in A\}$. The map f' is defined by $f'(a') = (a', 0) + S$ and β is given by $\beta(b) = (0, b) + S$. Moreover, g' is defined by $g'((a', b) + S) = g(b)$. Thus,

$$\begin{aligned}
\ker(g') &= \{(a', b) + S : g(b) = 0\} = \{(a', b) + S : b \in \ker(g) = \mathrm{im}(f)\} \\
&= \{(a', f(a)) : a' \in A, a \in A\} = \{(a' + \alpha(a), 0) + S : a' \in A', a \in A\} \\
&= f'(A').
\end{aligned}$$

Thus, $\ker(g') = \mathrm{im}(f')$. Finally, we have $\ker(f') = \{a' \in A' : (a', 0) \in S\}$. If $(a', 0) \in S$, then there is an $a \in A$ with $(a', 0) = (\alpha(a), -f(a))$. From this we conclude that $f(a) = 0$, so $a = 0$. Then $a' = \alpha(0) = 0$.

To determine the cocycle z' associated to $\mathcal{E}'$, let $l : C \to B$ with $g \circ l = \mathrm{id}_C$. Then we obtain a cocycle z representing $\mathcal{E}$ by the formula $z(c_1, c_2) = f^{-1}\left(l(c_1) + l(c_2) - l(c_1 + c_2)\right)$. To obtain a cocycle for $\mathcal{E}'$, we first need a map $l' : C \to B'$ with $g' \circ l' = \mathrm{id}_C$. Set $l' = \beta \circ l$. Commutativity of the diagram above shows that this is a valid choice. If z' is the corresponding cocycle, then

$$\begin{aligned}
z'(c_1, c_2) &= (f')^{-1}\left(l'(c_1) + l'(c_2) - l'(c_1 + c_2)\right) \\
&= (f')^{-1}\left(\beta l(c_1) + \beta l(c_2) - \beta l(c_1 + c_2)\right) \\
&= (f')^{-1}\left(\beta f(z(c_1, c_2))\right) = (f')^{-1}(f'\alpha(z(c_1, c_2))) \\
&= \alpha(z(c_1, c_2)).
\end{aligned}$$

since β is a homomorphism. Thus, $z' = \alpha \circ z$. To finish the argument, we note that the calculation to show $z' = \alpha \circ z$ did not need anything other than the fact that we have a map $(\alpha, \beta, \mathrm{id}_C) : \mathcal{E} \to \mathcal{E}'$. Thus, if we have another extension $\widetilde{\mathcal{E}}$ and a map $(\alpha, \beta', \mathrm{id}_C) : \mathcal{E} \to \widetilde{\mathcal{E}}$, then $\alpha \circ z$ also represents $\widetilde{\mathcal{E}}$. Since $\widetilde{\mathcal{E}}$ and $\mathcal{E}'$ are represented by the same cocycle class, they are equivalent extensions. $\square$

Lemma 3.3. *Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an extension and let $\gamma : C' \to C$ be a map. Then there is an extension $\mathcal{E}' : 0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$, unique up to equivalence, and a map $(\mathrm{id}_A, \beta, \gamma) : \mathcal{E} \to \mathcal{E}'$. Furthermore, if z is a cocycle representing $\mathcal{E}$, then $z \circ (\gamma \times \gamma)$ is a cocycle representing $\mathcal{E}'$.*

Proof. The ideas will be similar to the proof of the previous lemma, so we give fewer details. Let B' be the pullback of the diagram

$$\begin{tikzcd}
 & C' \arrow[d, "\gamma"] \\
B \arrow[r, "g"] & C
\end{tikzcd}$$

with associated maps $g' : B' \to C'$ and $\beta : B' \to B$. From the zero map $A \to C$ and $f : A \to B$ the UMP yields a unique map $f' : A \to B'$ with $g' \circ f' = 0$ and $\beta \circ f' = f$. We then have a diagram

$$\begin{tikzcd}
 & A \arrow[r, "f'"] \arrow[d, "\mathrm{id}"] & B' \arrow[r, "g'"] \arrow[d, "\beta"] & C' \arrow[d, "\gamma"] & \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0.
\end{tikzcd}$$

Because the left square is commutative, f' is injective. We can show the top sequence yields an exact sequence $0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C' \to 0$ by a similar argument to that of the previous lemma; we leave out the details.

To determine the cocycle for $\mathcal{E}'$, let $l : C \to B$ with $g \circ l = \mathrm{id}_C$, and let z be the corresponding cocycle. Recalling that $B' = \{(b, c') \in B \oplus C' : g(b) = \gamma(c')\}$, we produce a map $l' : C' \to B'$ with $g' \circ l' = \mathrm{id}_{C'}$ by defining $l'(c') = (l\gamma(c'), c')$. Recalling the definition of

g', we have $g'l'(c') = g'(l\gamma(c'), c')) = c'$. Thus, $g' \circ l' = \mathrm{id}$. The cocycle z' corresponding to this choice of l' is

$$\begin{aligned} z'(c'_1, c'_2) &= (f')^{-1}\left(l'(c'_1) + l'(c'_2) - l'(c'_1 + c'_2)\right) \\ &= (f')^{-1}\left((l\gamma(c'_1), c'_1) - (l\gamma(c'_2), c'_2) - (l\gamma(c'_1 + c'_2), c'_1 + c'_2)\right) \\ &= (f')^{-1}\left((l\gamma(c'_1) + l\gamma(c'_2) - l\gamma(c'_1 + c'_2), 0)\right) \\ &= (f')^{-1}\left(fz(\gamma(c'_1), \gamma(c'_2), 0) = z(\gamma(c'_1), \gamma(c'_2)\right) \end{aligned}$$

Since $f'(a) = (f(a), 0)$.Therefore, $z' = z \circ (\gamma \times \gamma)$, as desired.

Finally, we show that $\mathcal{E}'$ is unique, up to equivalence. Suppose there is an extension $\widetilde{\mathcal{E}} : 0 \to A \xrightarrow{\widetilde{f}} \widetilde{B} \xrightarrow{\widetilde{g}} C' \to 0$ and a map $(\mathrm{id}_C, \widetilde{\beta}, \widetilde{\gamma}) : \widetilde{\mathcal{E}} \to \mathcal{E}$. This map corresponds to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\widetilde{f}"] \arrow[d, "\mathrm{id}"] & \widetilde{B} \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{\beta}"] & C' \arrow[r] \arrow[d, "\widetilde{\gamma}"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0. \end{tikzcd}$$

We have the commutative diagram

$$\begin{tikzcd} \widetilde{B} \arrow[drr, bend left, "\widetilde{g}"] \arrow[ddr, bend right, "\widetilde{\beta}"'] & & \\ & B' \arrow[r, "g'"] \arrow[d, "\beta"] & C' \arrow[d, "\gamma"] \\ & B \arrow[r, "g"] & C. \end{tikzcd}$$

Since $\gamma \circ \widetilde{g} = g \circ \widetilde{\beta}$, the UMP for pullbacks gives a map $h : \widetilde{B} \to B'$ such that $g' \circ h = \widetilde{g}$ and $\beta \circ h = \widetilde{\beta}$. We then have a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\widetilde{f}"] \arrow[d, "\mathrm{id}"] & \widetilde{B} \arrow[r, "\widetilde{g}"] \arrow[d, "h"] & C' \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f'"] \arrow[d, "\mathrm{id}"] & B' \arrow[r, "g'"] \arrow[d, "\beta"] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

where we know that every square commutes except for perhaps to the top left corner. We need to show that $f' = h \circ \widetilde{f}$ to see that it does commute. However, the uniqueness part of the UMP for pullbacks will give this to us. We have maps $f : A \to B$ and $0 : A \to C'$ such

that $g \circ f = 0 = \gamma \circ 0$, yielding the following commutative diagram

$$\begin{tikzcd}
A \arrow[dr, "h\widetilde{f}"] \arrow[dr, "f'"'] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "f"'] & & \\
& B' \arrow[r, "g'"] \arrow[d, "\beta"] & C' \arrow[d, "\gamma"] \\
& B \arrow[r, "g"] & C.
\end{tikzcd}$$

Since $f', h\widetilde{f} : A \to B'$, the UMP for pullbacks will show that they will be equal if $g'f' = 0 = g'h\widetilde{f}$ and $\beta f' = f = \beta h\widetilde{f}$. However, all of these equations hold from commutativity of the various diagrams above. So, we do have a map $(\mathrm{id}_A, h, \mathrm{id}_C) : \widetilde{\mathcal{E}} \to \mathcal{E}'$; therefore, these extensions are equivalent. $\square$

Let $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and $\mathcal{E}' : 0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C \to 0$ be two extensions of A by C. To construct the Baer sum of two extensions, we first consider the direct sum

$$0 \to A \oplus A \xrightarrow{f \oplus f'} B \oplus B' \xrightarrow{g \oplus g'} C \oplus C \to 0$$

of the extensions. It is easy to see that this sequence is exact. Define homomorphisms $\nabla : A \oplus A \to A$ and $\Delta : C \to C \oplus C$ by $\nabla(a_1, a_2) = a_1 + a_2$ and $\Delta(c) = (c, c)$. By the two lemmas, we have two extensions $\mathcal{E}_1 : 0 \to A \to E_1 \to C \oplus C \to 0$ and $\mathcal{E}_2 : 0 \to A \to E_2 \to C \to 0$ such that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \oplus A \arrow[r, "f \oplus f'"] \arrow[d, "\nabla"] & B \oplus B' \arrow[r, "g \oplus g'"] \arrow[d, "\beta_1"] & C \oplus C \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A \arrow[r, "f_1"] & E_1 \arrow[r, "g_1"] & C \oplus C \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "f_2"] \arrow[u, "\mathrm{id}"] & E_2 \arrow[r, "g_2"] \arrow[u, "\beta_2"] & C \arrow[r] \arrow[u, "\Delta"] & 0
\end{tikzcd}$$

commutes. We call the extension $\mathcal{E}_2$ the *Baer Sum* of $\mathcal{E}$ and $\mathcal{E}'$. We denote it symbolically by $\mathcal{E} + \mathcal{E}'$.

Proposition 3.4. *Let : $\mathcal{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and $\mathcal{E}' : 0 \to A \xrightarrow{f'} B' \xrightarrow{g'} C \to 0$ be extensions of A by C. If z and z' are cocycles for these extensions, respectively, then $z + z'$ is a cocycle for $\mathcal{E} + \mathcal{E}'$.*

Proof. Let z and z' be the cocycles for $\mathcal{E}$ and $\mathcal{E}'$, respectively. We recall how to define them. We choose functions $l : C \to B$ and $l' : C \to B'$ such that $g \circ l = \mathrm{id}_C$ and $g' \circ l' = \mathrm{id}_C$. Then

$$\begin{aligned} z(c_1, c_2) &= f^{-1}\left(l(c_1) + l(c_2) - l(c_1 + c_2)\right), \\ z'(c_1, c_2) &= (f')^{-1}\left(l'(c_1) + l'(c_2) - l'(c_1 + c_2)\right). \end{aligned}$$

To have similar data for the direct sum extension, we define $l \oplus l' : C \oplus C \to B \oplus B'$ by

$$(l \oplus l')\,((c_1, c_2), (c_1', c_2')) = (l(c_1, c_2), l'(c_1', c_2')).$$

It is easy to check that $(g \oplus g') \circ (l \oplus l') = \mathrm{id}_{C \oplus C}$. Furthermore, the cocycle arising from this choice, which we denote by $z \oplus z'$, is given by

$$(z \oplus z')\,((c_1, c_2), (c_1', c_2')) = (z(c_1, c_1'), z'(c_2, c_2')).$$

To prove the proposition we need to write out the cocycle for the extension $\mathcal{E}+\mathcal{E}'$. We combine the results of the two lemmas to do this. Since $z \oplus z'$ is the cocycle for the top extension, $\nabla \circ (z \oplus z')$ is the cocycle for the middle extension, and then $y = (\nabla \circ (z \oplus z')) \circ (\Delta \oplus \Delta)$ is the cocycle for the bottom extension. To be more explicit, if $c_1, c_2 \in C$, then

$$\begin{aligned} y(c_1, c_2) &= \nabla\,(z \oplus z')((c_1, c_1), (c_2, c_2)) = \nabla\,(z(c_1, c_2), z'(c_1, c_2)) \\ &= z(c_1, c_2) + z'(c_1, c_2). \end{aligned}$$

Thus, $z + z'$ is a cocycle representing our extension $\mathcal{E} + \mathcal{E}'$. □

The proposition then tells us that addition in $\mathrm{Ext}(C, A)$ corresponds to Baer sum of extensions, via the identification of an element of $\mathrm{Ext}(C, A)$ with an extension of A by C.

4 The Ext Functors

Just as Hom yields a class of functors, so does Ext. We describe these functors now. For each Abelian group M we will obtain two functors $\mathrm{Ext}(M, -)$ and $\mathrm{Ext}(-, M)$. The first is covariant and the second contravariant. We know what these functors do to objects, since they send a group A to $\mathrm{Ext}(M, A)$ and $\mathrm{Ext}(A, M)$, respectively. We must define their action on maps. Fix an Abelian group M. If $\alpha : A \to A'$ is a homomorphism, we define $\alpha_* : \mathrm{Ext}(M, A) \to \mathrm{Ext}(M, A')$ by

$$\alpha_*(z + B(M, A)) = \alpha \circ z + B(M, A').$$

Similarly, we define $\alpha^* : \mathrm{Ext}(A', M) \to \mathrm{Ext}(A, M)$ by

$$\alpha^*(z + B(A', M)) = z \circ (\alpha \times \alpha) + B(A, M).$$

To help to understand these definitions, we note that these formulas arise from Lemmas 3.2 and 3.3. In terms of extensions, if $\mathcal{E} : 0 \to A \to B \to M \to 0$ is an extension of C by A, then $\alpha_*(\mathcal{E}) : 0 \to A' \to B' \to M \to 0$ is the unique up to equivalence extension making the following diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & M \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & M \arrow[r] & 0 \end{tikzcd}$$

A similar diagram helps to understand α^*.

Lemma 4.1. $\mathrm{Ext}(M, -)$ *is a covariant functor and* $\mathrm{Ext}(-, M)$ *is a contravariant functor.*

Proof. We prove only the first statement; the proof of the second is similar. Let $z \in Z(M, A)$, and let $\alpha : A \to A'$ be a homomorphism. We show that $\alpha \circ z$ is a symmetric cocycle. First, $\alpha z(m, 0) = \alpha(z(m, 0)) = \alpha(0) = 0$ and $\alpha z(0, m) = \alpha(z(0, m)) = 0$. Second, $\alpha z(m, n) = \alpha(z(m, n)) = \alpha(z(m, n)) = \alpha z(m, n)$. Finally, to check the cocycle condition, let $m, n, p \in M$. Then $z(m, n) + z(m + n, p) = z(n, p) + z(m, n + p)$. Applying α to this equation and remembering that α is a group homomorphism gives $\alpha z(m, n) + \alpha z(m + n, p) = \alpha z(n, p) + \alpha z(m, n + p)$. Thus, $\alpha \circ z$ is a cocycle. If z is a coboundary, then there is a function $l : M \to A$ with $f(m, n) = l(m) + l(n) - l(m + n)$ for all $m, n \in M$. Then $\alpha z(m, n) = \alpha l(m) + \alpha l(n) - \alpha l(m + n)$. Thus, as $\alpha \circ l : M \to C$, we see that αz is a coboundary. Therefore, α_* is well-defined. Because α is a homomorphism and addition in $Z(M, A)$ is pointwise, it is clear that α_* is a homomorphism.

To show that $\mathrm{Ext}(M, -)$ is a functor, we have to show that it behaves correctly on maps. First, consider the identity $\mathrm{id} : A \to A$. Then $\mathrm{id}_* : \mathrm{Ext}(M, A) \to \mathrm{Ext}(M, A)$ is defined by $z + B(M, A) \mapsto \mathrm{id} \circ z + B(M, A) = z + B(M, A)$. Thus, id_* is the identity on $\mathrm{Ext}(M, A)$. Next, let $\alpha : A \to C$ and $\beta : C \to D$ be homomorphisms. Then

$$\begin{aligned}(\beta\alpha)_*(z + B(M, A)) &= \beta\alpha z + B(M, D) = \beta_*(\alpha z + B(M, C)) \\ &= \beta_*(\alpha_*(z + B(M, A)).\end{aligned}$$

Therefore, $(\beta\alpha)_* = \beta_*\alpha_*$. Finally, if $\alpha, \beta : A \to C$ are homomorphisms, then

$$\begin{aligned}(\alpha + \beta)_*(z + B(M, A)) &= (\alpha + \beta) \circ z + B(M, C) = (\alpha z + \beta z) + B(M, C) \\ &= \alpha_*(z + B(M, C)) + \beta_*(z + B(M, C)).\end{aligned}$$

Thus, $(\alpha + \beta)_* = \alpha_* + \beta_*$. This finishes the proof that $\mathrm{Ext}(M, -)$ is a covariant functor. $\square$

Our goal is, for each exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ and each group M, to produce an exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}(M, A) &\xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C) \xrightarrow{\delta} \\ \mathrm{Ext}(M, A) &\xrightarrow{f_*} \mathrm{Ext}(M, B) \xrightarrow{g_*} \mathrm{Ext}(M, C) \to 0\end{aligned}$$

and an exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}(C, M) &\xrightarrow{g^*} \mathrm{Hom}(B, M) \xrightarrow{f^*} \mathrm{Hom}(A, M) \xrightarrow{\delta} \\ \mathrm{Ext}(C, M) &\xrightarrow{g^*} \mathrm{Ext}(B, M) \xrightarrow{f^*} \mathrm{Ext}(A, M) \to 0.\end{aligned}$$

We need to define the *connecting homomorphisms* δ. We define $\delta : \mathrm{Hom}(M, C) \to \mathrm{Ext}(M, A)$ as follows. Let $\sigma \in \mathrm{Hom}(M, C)$. If z is the cocycle associated to the exact sequence, we define $\delta(\sigma) = z \circ (\sigma \times \sigma) + B(M, A)$. In other words, $\delta(\sigma)$ is represented by the cocycle that sends (m, n) to $z(\sigma(m), \sigma(n))$. Similarly, we define $\delta : \mathrm{Hom}(A, M) \to \mathrm{Ext}(C, M)$ by $\delta(\sigma) = \sigma \circ z + B(C, M)$.

Theorem 4.2. *Let* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *be an exact sequence of Abelian groups. For every Abelian group* M*, the sequences*

$$\begin{aligned} 0 \to \operatorname{Hom}(M,A) \xrightarrow{f_*} \operatorname{Hom}(M,B) \xrightarrow{g_*} \operatorname{Hom}(M,C) \xrightarrow{\delta} \\ \operatorname{Ext}(M,A) \xrightarrow{f_*} \operatorname{Ext}(M,B) \xrightarrow{g_*} \operatorname{Ext}(M,C) \to 0 \end{aligned}$$

and an exact sequence

$$\begin{aligned} 0 \to \operatorname{Hom}(C,M) \xrightarrow{g^*} \operatorname{Hom}(B,M) \xrightarrow{f^*} \operatorname{Hom}(A,M) \xrightarrow{\delta} \\ \operatorname{Ext}(C,M) \xrightarrow{g^*} \operatorname{Ext}(B,M) \xrightarrow{f^*} \operatorname{Ext}(A,M) \to 0 \end{aligned}$$

are exact.

Proof. We prove the exactness of the second sequence and leave the proof of the first sequence to the reader. Let z be a cocycle representing the exact sequence. We first show that the composition of any pair of consecutive maps is 0. For the pair of $g^*$and f^*, we have $f^* \circ g^* = (g \circ f)^* = 0^* = 0$ since $\operatorname{Hom}(-,M)$ and $\operatorname{Ext}(-,M)$ is a contravariant functor. Next, to see that $\delta \circ f^* = 0$, take $\sigma \in \operatorname{Hom}(B,M)$. Then $\delta(f^*(\sigma)) = \delta(\sigma \circ f) = \sigma f z$. We recall that $z(c_1,c_2) = f^{-1}\left(l(c_1) + l(c_2) - l(c_1+c_2)\right)$ for an appropriate function $l : C \to B$. Then $\sigma f z(c_1,c_2) = \sigma l(c_1) + \sigma l(c_2) - \sigma l(c_1+c_2)$. Since $\sigma l : C \to M$, we see that $\sigma f z$ satisfies the definition of a coboundary. Thus, $(\delta \circ f)(z) = 0$ in $\operatorname{Ext} C, M)$.

Next, we see that $g^* \circ \delta = 0$. Take $\sigma \in \operatorname{Hom}(A,M)$. Then $g^*(\delta(\sigma))$ is the class of the cocycle y, where

$$y(b_1,b_2) = \sigma f^{-1}\left(lg(b_1) + lg(b_2) - lg(b_1+b_2)\right).$$

The function l satisfies $g \circ l = \mathrm{id}_C$. Therefore, for each $b \in B$ there is an $a \in A$ with $lg(b) - b = f(a)$. Consequently, there is a function $t : B \to A$ such that $lg(b) - b = f(t(b))$. The cocycle y can then be represented as

$$\begin{aligned} y(b_1,b_2) &= \sigma f^{-1}\left(ft(b_1) + ft(b_2) - ft(b_1+b_2)\right) \\ &= \sigma t(b_1) + \sigma t(b_2) - \sigma t(b_1+b_2). \end{aligned}$$

The function $\sigma t : B \to M$, and so this is the definition for y to be a coboundary. Thus, $y = 0$ in $\operatorname{Ext}(B,M)$. Therefore, $g^* \circ \delta = 0$.

At this point we have verified that the sequence is a zero sequence. Since Hom is right exact, $0 \to \operatorname{Hom}(C,M) \xrightarrow{g^*} \operatorname{Hom}(B,M) \xrightarrow{f^*} \operatorname{Hom}(A,M)$ is exact. To prove exactness, we then have to prove exactness at the final Hom term and the three Ext terms.

$\ker(\delta) = \operatorname{im}(f^*)$: Take $\sigma \in \operatorname{Hom}(A,M)$ with $\delta(\sigma) = 0$. The extension corresponding to $\delta(\sigma)$ is then split exact. By Lemma 3.2, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & M \arrow[r, "i"] & B \oplus C \arrow[r, "\pi"] & C \arrow[r] & 0. \end{tikzcd}$$

Let $j : M \oplus C \to M$ be the canonical projection. Set $\tau = j \circ \beta \in \mathrm{Hom}(B, M)$. We have $\tau f = j\beta f = ji\sigma = \sigma$, so $\sigma = f^*(\tau)$. Thus, $\ker(\delta) = \mathrm{im}(f^*)$.

$\ker(g^*) = \mathrm{im}(\delta)$: Let $z' \in \mathrm{Ext}(C, M)$ such that $g^*(z') = 0$. By Lemma 3.3, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{E}: & 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "j\beta"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\mathcal{E}': & 0 \arrow[r] & M \arrow[r, "f'"] & E \arrow[r, "g'"] & C \arrow[r] & 0 \\
\mathcal{E}'\beta: & 0 \arrow[r] & M \arrow[r, "i"] \arrow[u, "\mathrm{id}"] & B \oplus M \arrow[r, "\pi"] \arrow[u, "\beta"] & B \arrow[r] \arrow[u, "g"] & 0.
\end{tikzcd}$$

Let $j : B \to B \oplus M$ be the canonical map. Then $g'\beta jf = g\pi jf = gf = 0$. Therefore, for each $a \in A$, there is an $m \in M$ with $g'\beta jf(a) = f'(m)$. Since f' is injective, m is unique. Thus, there is a map $\sigma : A \to M$ with $g'\beta jf = f'\sigma$. We thus have a commutative diagram

$$\begin{tikzcd}
\mathcal{E}: & 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\sigma"] & B \arrow[r, "g"] \arrow[d, "j\beta"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\mathcal{E}': & 0 \arrow[r] & M \arrow[r, "f'"] & E \arrow[r, "g'"] & C \arrow[r] & 0
\end{tikzcd}$$

By the description of the cocycle z' for the extension $\mathcal{E}'$, we have $z' = \sigma z = \delta(\sigma)$. Thus, $z' \in \mathrm{im}(\delta)$.

$\ker(f^*) = \mathrm{im}(g^*)$: Suppose that $0 \to M \xrightarrow{f'} E \xrightarrow{g'} B \to$ is an extension of M by B whose image extension in $\mathrm{Ext}(A, M)$ is 0. By Lemma 3.3, we construct this image extension by forming the pullback of the maps $f : A \to B$ and $g' : E \to B$. We have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f'"] & E \arrow[r, "g'"] & B \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "i"] \arrow[u, "\mathrm{id}"] & P \arrow[r, "\pi"] \arrow[u, "\beta"] & A \arrow[r] \arrow[u, "f"] & 0.
\end{tikzcd}$$

Since the bottom extension represents 0 in $\mathrm{Ext}(A, M)$, the sequence is split exact. Thus, there are maps $j : A \to P$ and $k : P \to M$ with $\pi j = \mathrm{id}_A$ and $ki = \mathrm{id}_M$. Let Q be the pushout of the maps $\mathrm{id} : M \to M$ and $f' : M \to E$. By the UMP for pushouts, there is a map $h : Q \to C$ with $h \circ \delta = g \circ g'$ and $h \circ t = 0$. Thus, the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "t"] & Q \arrow[r, "h"] & C \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r, "f'"] \arrow[u, "\mathrm{id}"] & E \arrow[r, "g'"] \arrow[u, "\delta"] & B \arrow[r] \arrow[u, "g"] & 0 \\
0 \arrow[r] & M \arrow[r, "i"] \arrow[u, "\mathrm{id}"] & P \arrow[r, "\pi"] \arrow[u, "\beta"] & A \arrow[r] \arrow[u, "f"] & 0.
\end{tikzcd}$$

If we show that the top row is exact, then it will represent an element of $\mathrm{Ext}(C, M)$ mapping to the given element of $\mathrm{Ext}(B, M)$. We recall that $Q = (M \oplus E)/S$, where $S = \{(m, -f'(m) : m \in M\}$. By definition of h, the sequence is a zero sequence. The map t is injective, since $t(m) = (m, 0) + S$, and $(m, 0) \in S$ if and only if $m = 0$. Also, h is surjective, since if $c \in C$, then $c = g(b)$ for some $b \in B$, and so $c = gg'(e)$ for some e. Then $h\left((0, e) + S\right) = c$. Finally, we show that $\ker(h) = \mathrm{im}(t)$. Suppose that $h\left((m, e) + S\right) = 0$. Then $gg'(e) = 0$. So, $g'(e) = f(a)$ for some $a \in A$. But, $f(a) = g'\beta j(a)$. Set $p = j(a)$. Then $\beta(p) = f'k(p)$. Consequently, $g'(e) = g'f'k(p) = 0$. So, $e = f'(m')$ for some m', and so $(m, e) + S = (m, f'(m') + S = (m - m', 0) + S = t(m - m')$, as desired.

The hardest step is to show that f^* is surjective. For this we need to use that we are in the category of Abelian groups and not a category of modules; the fact we need is that subgroups of free Abelian groups are free Abelian. Let y be a cocycle representing a class in $\mathrm{Ext}(M, A)$, and let $0 \to M \xrightarrow{\sigma} E \xrightarrow{\tau} A \to 0$ be the corresponding extension. We then obtain an exact sequence $0 \to M \xrightarrow{\sigma} E \xrightarrow{f\tau} B \xrightarrow{g} C \to 0$. There is a free Abelian group F and an exact sequence $0 \to K \xrightarrow{\gamma} F \xrightarrow{\varepsilon} C \to 0$. Thus, K is also free Abelian, and so both K and F are projective. Therefore, there is a homomorphism $\mu : F \to B$ with $g \circ \mu = \varepsilon$. Furthermore, $g\mu\gamma = \varepsilon\gamma = 0$; thus, $\mu(\gamma(K)) \subseteq \ker(g) = \mathrm{im}(f\tau)$. Thus, since K is projective, there is a homomorphism $\nu : K \to E$ with $f\tau\nu = \mu\gamma$. We then have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] & K \arrow[r, "\gamma"] \arrow[d, "\nu"] & F \arrow[r, "\epsilon"] \arrow[d, "\mu"] & C \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & M \arrow[r, "\sigma"] & E \arrow[r, "f\tau"] & B \arrow[r, "g"] & C \arrow[r] & 0.
\end{tikzcd}$$

Let E' be the pushout of the diagram

$$\begin{tikzcd}
K \arrow[r, "\gamma"] \arrow[d, "\nu"] & F \\
E &
\end{tikzcd}$$

We then have a map $h : E' \to B$ such that the following diagram commutes

$$\begin{tikzcd}
 & K \arrow[r, "\gamma"] \arrow[d, "\nu"] & F \arrow[d, "\mu'"] \arrow[ddr, bend left, "\mu"] & \\
M \arrow[r, "\sigma"] & E \arrow[r, "\theta"] \arrow[drr, bend right, "f\tau"'] & E' \arrow[dr, "h"] & \\
 & & & B.
\end{tikzcd}$$

We claim that $0 \to M \xrightarrow{\theta\sigma} E' \xrightarrow{h} B \to 0$ is an extension of M by B whose image in $\mathrm{Ext}(M, B)$ maps to the given extension in $\mathrm{Ext}(M, A)$. We have several things to verify. First, we see that $h\theta\sigma = f\tau\sigma = 0$ by the exactness of the 6-term sequence above. Next, we show that h is surjective. To do this, let $b \in B$. Then $g(b) = \varepsilon(x)$ for some $x \in F$. Then

$g\mu(x) = \varepsilon(x) = g(b)$. Thus, $\mu(x) = b + f(a)$ for some $a \in A$. If we write $a = \tau(e)$, then $h\theta(e) = f\tau(e) = f(a)$. Thus,

$$b = \mu(x) - f(a) = h\mu'(x) - h\theta(e) \in \operatorname{im}(h),$$

as desired. To finish the proof that the sequence $0 \to M \xrightarrow{\theta\sigma} E' \xrightarrow{h} B \to 0$ is exact, we show that $\ker(h) = \operatorname{im}(\theta\sigma)$. Recall that $E' = (E \oplus F)/S$ with $S = \{(\nu(k), -\gamma(k)) : k \in K\}$. Suppose that $h((e,x) + S) = 0$. Then by definition of h, we have $f\tau(e) + \mu(z) = 0$. Applying g yields $0 = g\mu(f) = \varepsilon(z)$. Thus, since $0 \to K \to F \to C \to 0$ is exact, $z = \gamma(k)$ for some $k \in K$. Then

$$f\tau(e) = -\mu(z) = -\mu\gamma(k) = -f\tau\nu(k).$$

This yields $f\tau(e + \nu(k)) = 0$, so $e + \nu(k) = \sigma(m)$ for some $m \in M$. Finally, we see that

$$\begin{aligned}(e,x) + S &= (e + \nu(k), x - \gamma(k)) + S \\ &= (\sigma(m), 0) + S = \theta\sigma(m).\end{aligned}$$

This finishes the proof that the sequence is exact. To show that this sequence maps onto the given one, we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\sigma"] \arrow[d, "\mathrm{id}"] & E \arrow[r, "\tau"] \arrow[d, "\theta"] & A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & M \arrow[r, "\theta\sigma"] & E' \arrow[r, "h"] & B \arrow[r] & 0
\end{tikzcd}$$

which clearly commutes. If y' is a cocycle representing the bottom extension, then $\alpha^*(y') = y$; this follows from Lemma 3.3. This proves that f^* is surjective. $\square$

Example 4.3. Let p be a positive integer. We compute the group $\operatorname{Ext}(\mathbb{Z}_p, \mathbb{Z}_p)$; in fact, we determine $\operatorname{Ext}(\mathbb{Z}_p, B)$ for any group B. Consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_p \to 0$, where the map $\mathbb{Z} \to \mathbb{Z}$ is multiplication by p. Recall that, since $\mathbb{Z}$ is free Abelian, $\operatorname{Ext}(\mathbb{Z}, B) = 0$. Applying the theorem to B, we have the exact sequence

$$\begin{gathered}0 \to \operatorname{Hom}(\mathbb{Z}_p, B) \xrightarrow{\pi^*} \operatorname{Hom}(\mathbb{Z}, B) \xrightarrow{p^*} \operatorname{Hom}(\mathbb{Z}, B) \xrightarrow{\delta} \\ \operatorname{Ext}(\mathbb{Z}_p, B) \xrightarrow{\pi^*} \operatorname{Ext}(\mathbb{Z}, B) \xrightarrow{p^*} \operatorname{Ext}(\mathbb{Z}, M) \to 0.\end{gathered}$$

Recall that $\operatorname{Hom}(\mathbb{Z}, B) \cong B$ via the map $\sigma \mapsto \sigma(1)$. Also, $\operatorname{Hom}(\mathbb{Z}_p, B) \cong {}_pB$, the p-torsion subgroup of B, which is defined by ${}_pB = \{b \in B : pb = 0\}$. This isomorphism is given by $\sigma \mapsto \sigma(\overline{1})$. A short computation, using the isomorphism $\operatorname{Hom}(\mathbb{Z}, B) \cong B$ shows that p^* translates to the multiplication by p map on B. The long exact sequence above then reduces to

$$0 \to {}_pB \to B \xrightarrow{p} B \xrightarrow{\delta} \operatorname{Ext}(\mathbb{Z}_p, B) \to 0.$$

Therefore, $\operatorname{Ext}(\mathbb{Z}_p, B)$ is the cokernel p^*, so $\operatorname{Ext}(\mathbb{Z}_p, B) \cong B/pB$. In particular, $\operatorname{Ext}(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p \cong \operatorname{Ext}(\mathbb{Z}_p, \mathbb{Z})$. As we saw in an example earlier, applying $\operatorname{Hom}(-, \mathbb{Z}_p)$ to the sequence $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}_p \to 0$ ruins exactness; we obtain the sequence $0 \to 0 \to 0 \to \mathbb{Z}_p$. By using the Ext groups, we fill it out to the exact sequence $0 \to 0 \to 0 \to \mathbb{Z}_p \xrightarrow{\delta} \mathbb{Z}_p \to 0 \to 0$.

5 Homological Algebra view of Ext

We give a brief description of how Ext groups arise in the larger context of homological algebra. This section assumes some knowledge of chain complexes. Let A and M be Abelian groups. For $n \geq 0$ let $A_n = \mathrm{Map}(A^{n+1}, M)$ be the set of all functions $A^{n+1} \to M$. By using pointwise addition, we view A_n as an Abelian group. If F_n is the free Abelian group on the set A^{n+1}, then the UMP for free Abelian groups implies that $A_n \cong \mathrm{Hom}(F_n, M)$. We make $\{A_n\}$ into a cochain complex by defining $d_{n-1} : A_{n-1} \to A_n$ by

$$\begin{aligned} d_{n-1}(f)(a_0, \ldots, a_n) = f(a_1, \ldots, a_n) &+ \sum_{i=1}^{n-1} (-1)^i f(a_0, \ldots, a_i + a_{i+1}, \ldots, a_n) \\ &+ (-1)^n f(a_0, \ldots, a_{n-1}). \end{aligned}$$

For some notation, we write $A_{-1} = 0$ and $d_{-1} = 0$. By a boring calculation, one sees that $d_n \circ d_{n-1} = 0$ for all n. We then define $\mathrm{Ext}^n(A, M)$ to be the homology at the n-th stage of this complex. In other words,

$$\mathrm{Ext}^n(A, M) = \ker(d_n)/\operatorname{im}(d_{n-1}).$$

For $n = 0$, we have $\mathrm{Ext}^0(A, M) = \ker(d_0)/\operatorname{im}(d_{-1}) = \ker(d_0)$ since $d_{-1} = 0$. As a special case of d_n, we see that $d_0(f)(a_0, a_1) = f(a_1) - f(a_0 + a_1) + f(a_1)$. Thus, $f \in \ker(d_0)$ if and only if f is a homomorphism, and $d_0(f) \in \operatorname{im}(d_0)$ is a 2-coboundary. Moreover, if $f : A \times A \to M$, then $d_1(f) = 0$ if and only if, for all $a_0, a_1, a_2 \in A$, we have

$$f(a_1, a_2) - f(a_0 + a_1, a_2) + f(a_0, a_1 + a_2) - f(a_0, a_1) = 0.$$

Thus, $d_1(f) = 0$ if and only if f is a 2-cocycle. Thus, we see that $\mathrm{Ext}^1(A, M) = \mathrm{Ext}(A, M)$ and $\mathrm{Ext}^0(A, M) = \mathrm{Hom}(A, M)$.

An alternative, more abstract view of this situation is via free resolutions. If $F : \cdots \to F_n \to \cdots \to F_1 \to A \to 0$ is a free resolution of A; that is, an exact sequence with each F_i a free Abelian group, then $\mathrm{Ext}^n(A, M)$ is the n-th homology group of the complex $\{\mathrm{Hom}(F, M)\}$. In other words, we consider the cochain complex

$$\mathrm{Hom}(F_1, M) \to \mathrm{Hom}(F_2, M) \to \cdots$$

and take homology of the complex. Note that the maps changed direction since $\mathrm{Hom}(-, M)$ is contravariant. The F_i defined above provide a free resolution of A, since we have the complex

$$\cdots \to F_n \to \cdots \to F_1 \to F_0 \to A \to 0$$

defined as follows: the map $d_0 : F_0 \to A$ is the map arising from the UMP for free Abelian groups and from the identity map $A \to A$; recall that F_0 is the free Abelian group on A. If $n \geq 1$, then $d_n : F_n \to F_{n-1}$ arises from the UMP and the map $A^{n+1} \to A^n \subseteq F_{n-1}$ given by

$$(a_0, \ldots, a_n) \mapsto \sum_{i=0}^{n} (-1)^i (a_0, \ldots, a_i + a_{i+1}, \ldots, a_n),$$

One gets the maps $A_n = \mathrm{Hom}(F_n, M) \to A_{n+1} = \mathrm{Hom}(F_{n+1}, M)$ by applying the functor $\mathrm{Hom}(-, M)$ to the map $F_{n+1} \to F_n$. It is a theorem of homological algebra that the homology groups one gets do not depend on the choice of free resolution of A.

If A is an Abelian group and F is a free Abelian group such that there is an exact sequence $0 \to K \to F \to A \to 0$, then K is also free Abelian. Thus, we can take this to be a free resolution. If we apply the machinery to this extension, then we see that all homology groups for $n \geq 2$ are 0. This implies that $\mathrm{Ext}^n(A, M) = 0$ if $n \geq 2$. Therefore, it was sufficient for us to define $\mathrm{Ext} = \mathrm{Ext}^1$.

As a final note, if R is a ring, we could define groups $\mathrm{Ext}_R(A, M)$ for any two R-modules A and M, by considering the problem of extensions in the category of R-modules. In fact, we can define a sequence $\{\mathrm{Ext}^n_R(A, M) : n \in \mathbb{N}\}$ of groups, for which $\mathrm{Ext}^0_R(A, M) = \mathrm{Hom}_R(A, M)$ and $\mathrm{Ext}^1_R(A, M) = \mathrm{Ext}_R(A, M)$.