Homework Exercises 1

September 3, 2002

1. A subgroup $H$ of a group $G$ is said to be a characteristic subgroup if $\varphi(H) \subseteq H$ for every automorphism $\varphi$ of $G$.

(a) Prove that a characteristic subgroup is normal.
(b) Prove that the commutator subgroup of a group is characteristic.
(c) Prove that the center of a group is characteristic.
(d) Prove that every subgroup of a cyclic group is characteristic.
(e) Prove that $A_n$ is a characteristic subgroup of $S_n$. You may assume that $n \geq 5$ even though the result is true for every $n$.
(f) Let $H = \{e, (12)(34), (13)(24), (14)(23)\}$. Prove that $H$ is a characteristic subgroup of $S_4$.
(g) Prove that no nontrivial subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is characteristic.
(h) If $H$ is a characteristic subgroup of $K$ and $K$ is a characteristic subgroup of $G$, prove that $H$ is a characteristic subgroup of $G$. Find an example to show that this statement is false if characteristic is replaced by normal.
(i) Let $G$ be a finite group. If $H$ is the unique subgroup of $G$ of a given order, prove that $H$ is a characteristic subgroup of $G$. Conclude that if $P$ is a normal $p$-Sylow subgroup of a finite group $G$, then $P$ is characteristic.

2. If $p$ and $q$ are distinct prime numbers, prove that $\text{Aut}(\mathbb{Z}_p \times \mathbb{Z}_q) \cong \text{Aut}(\mathbb{Z}_p) \times \text{Aut}(\mathbb{Z}_q)$.

3. If $G_1$ and $G_2$ are finite groups with relatively prime orders, prove that $\text{Aut}(G_1 \times G_2) \cong \text{Aut}(G_1) \times \text{Aut}(G_2)$.

4. Find examples of finite groups $G_1$ and $G_2$ with $\text{Aut}(G_1 \times G_2) \not\cong \text{Aut}(G_1) \times \text{Aut}(G_2)$.

5. If $n$ is a positive integer, prove that $\text{Aut}(\mathbb{Z}^n) \cong \text{Gl}_n(\mathbb{Z})$, the group of invertible $n \times n$ matrices over $\mathbb{Z}$, which is the group of all $n \times n$ matrices over $\mathbb{Z}$ with determinant $\pm 1$.

6. If $\varphi \in \text{Aut}(\mathbb{Q})$, prove that there is a nonzero rational number $\alpha$ so that $\varphi(x) = \alpha x$ for all $x \in \mathbb{Q}$. 

1
7. If \( \varphi \in \text{Aut}(\mathbb{R}) \) is continuous, prove that there is a nonzero real number \( \alpha \) so that 
\[ \varphi(x) = \alpha x \text{ for all } x \in \mathbb{R}. \]

8. Show that there exist non-continuous automorphisms of \( \mathbb{R} \).

(Hints: first prove that \( \mathbb{R} \times \mathbb{R} \cong \mathbb{R} \) as groups by considering them as \( \mathbb{Q} \)-vector spaces. Next, if \( \mathbb{Q} \subseteq \mathbb{Q}(T) \subseteq \mathbb{C} \) with \( T \) a set of algebraically independent elements over \( \mathbb{Q} \) and \( \mathbb{C} \) algebraic over \( \mathbb{Q}(T) \), produce lots of field automorphisms of \( \mathbb{C} \) by finding automorphisms of \( \mathbb{Q}(T) \), and then extending them to \( \mathbb{C} \) by the isomorphism extension theorem.)

9. Define \( \varphi_i, \sigma_i \in \text{Aut}(D_n) \) by \( \varphi_i(r) = r^i \) and \( \varphi_i(f) = f \), and \( \sigma_i(r) = r \) and \( \sigma_i(f) = r^i f \).

Show that \( \sigma_i \in \text{Aut}(D_n) \) for any \( i \) and \( \varphi_i \in \text{Aut}(D_n) \) if \( \gcd(i, n) = 1 \). Prove that any automorphism of \( D_n \) is of the form \( \sigma_j \circ \varphi_i \) for some \( i, j \). Prove that \( \varphi_i \circ \varphi_j = \varphi_{ij} \) and \( \sigma_i \circ \sigma_j = \sigma_{i+j} \). Prove that \( \{ \varphi_i : \gcd(i, n) = 1 \} \) is a subgroup isomorphic to \( \mathbb{Z}_n^* \) and \( \{ \sigma_i : i \in \mathbb{Z} \} \) is a subgroup isomorphic to \( \mathbb{Z}_n \). Prove the second group is normal in \( \text{Aut}(D_n) \).

(In fact, \( \text{Aut}(D_n) \cong \mathbb{Z}_n \rtimes_{\text{id}} \text{Aut}(\mathbb{Z}_n) \).)