Some Groups of Order 27

Let $G = \langle x, y \mid x^9 = y^9 = 1, xy^4 = y^7x, yx^4 = x^7y \rangle$. In this note we determine the structure of $G$. For our first observation, we note that, by the symmetry of the relations, there is an automorphism $\varphi$ of $G$ with $\varphi(x) = y$ and $\varphi(y) = x$. In other words, the automorphism of the free algebra $F(\{x, y\})$ that switches $x$ and $y$ sends the set of relations for $G$ to itself, which implies that we get the induced automorphism $\varphi$. One application of the existence of $\varphi$ is that the orders of $x$ and $y$ are the same, since $\varphi(x) = y$. Since $x^9 = 1$, the division algorithm shows that the order of $x$ is either 1, 3, or 9. We claim that $G$ is a non-Abelian group of order 27, whose center $Z(G)$ is cyclic of order 3 and for which $G/Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Our arguments will mostly consist of playing with the relations together with finding a homomorphic image of $G$ of order 27 inside $S_9$. We will start by producing the homomorphic image. Let $u$ and $v$ be the 9-cycles $(123456789)$ and $(168735492)$, respectively. Then a computation (by hand or computer) shows that $u$ and $v$ satisfy the relations for $G$. Thus, there is a homomorphism $\sigma$ from $G$ onto the subgroup $\langle u, v \rangle$ of $S_9$ that sends $x$ to $u$ and $y$ to $v$. Furthermore, the group $\langle u, v \rangle$ has order 27; this is most easily seen with a computer. This shows that $|G| \geq 27$. Moreover, since the order of $x$ and $y$ divides 9, and since the order of $u = \sigma(x)$ is 9 and must divide the order of $x$, we conclude that the order of $x$ (and of $y$) is equal to 9. To explain how to find the $u$ and $v$, in Maple one can get a permutation representation for a presented group, by picking a subgroup $H$ and considering the action of $G$ on $G/H$. If one picks $H$ so that $\bigcap_{g \in G} yHg^{-1} = 1$, then $G$ will be isomorphic to its image in $\text{Perm}(G/H)$. In this case with $H = \langle xy \rangle$, Maple yielded these values of $u$ and $v$. We will be able to conclude later that $H$ satisfies the desired condition; however, we do not need this.

We now show that $|G| = 27$ by some fairly involved computations with the relations for $G$. We start by obtaining simplified conjugacy relations. The equation $xy^4 = y^7x$ implies $xy^4x^{-1} = y^7$. Squaring both sides gives $xy^8x^{-1} = x^{14} = x^5$, and taking inverses of both sides yields $xyx^{-1} = y^{-5} = y^4$. From this we see that $x^2yx^{-2} = xy^4x^{-1} = (xyx^{-1})^4 = (y^4)^4 = y^{16} = y^7$. Similarly, $x^3yx^{-3} = y^4^3 = y^{64} = y$. Consequently, $x^3$ commutes with $y$; since it commutes with $x$, we see that $x^3 \in Z(G)$. Similarly, $y^3 \in Z(G)$. Moreover, from $yx = x^4y$ along with $x^9 = y^9 = 1$, we obtain $G = \{x^ry^s : 0 \leq r, s \leq 8\}$. This yields $|G| \leq 81$; however, this is not good enough. To do better, we first show that $Z(G) = \langle x^3, y^3 \rangle$. Suppose that $x^ny^m \in Z(G)$. Then $x^ny^m(x^n y^m)^{-1} = x$. By induction, we have $y^m xy^{-m} = x^4m$. Since $x^ny^m(x^n y^m)^{-1} = y^m xy = x^4m$ and the order of $x$ is 9, we see that $4m \equiv 1 \mod 9$. The order of 4 mod 9 is 3; this shows $m$ is divisible by 3. Similarly, $n$ is divisible by 3. Thus,
While this isn’t extremely simple, it does completely determine the multiplication. \( x^ny^m \in \langle x^3, y^3 \rangle \), as desired. Since \( x^3 \) and \( y^3 \) each have order 3, we get \( |Z(G)| = 1, 3, 9 \), depending on what relations there are between \( x^3 \) and \( y^3 \). Consider the quotient group \( G/Z(G) \). This group is generated by the cosets \( \pi \) and \( \eta \). We have \( \pi^3 = 1 \) and \( \eta^3 = 1 \) since \( x^3, y^3 \in Z(G) \). Furthermore, the relation \( xy^4 = y^7x \) yields \( \bar{xy}^4 = \bar{y}^7\pi \). Since \( \bar{y}^3 = 1 \), this simplifies to \( \bar{xy}^3 = \pi \). Thus, \( G/Z(G) \) is Abelian. Because it is generated by \( \pi \) and \( \eta \) and \( \pi^3 = \eta^3 = 1 \), the group \( G/Z(G) \) can only be \( 1, \mathbb{Z}_3 \), or \( \mathbb{Z}_3 \times \mathbb{Z}_3 \), up to isomorphism. However, the first two possibilities say that \( G/Z(G) \) is cyclic. If \( G/Z(G) \) is cyclic, then \( G \) is Abelian, which is false since \( G \) has a non-Abelian homomorphic image. Thus, \( G/Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \). Consequently, \( |G/Z(G)| = 9 \). We will determine \( |G| \) by finding \( |Z(G)| \).

We show \( Z(G) \) has order 3 by showing that \( x^3 = y^{-3} \). To see this, we have

\[
(xy)^3 = x(yx)yx = xx^4(y^2x)y = xx^4x^16y^2y = x^{21}y^3 = x^3y^3.
\]

Consequently, \( (xy)^3 \in Z(G) \). So, \( (xy)^4 = (xy)^3xy = (xy)^3y = xx^3y^3y = x^4y^4 \). Now, this can be rewritten as

\[
x^4yxx^{-1} = x^3x^2yxx^{-1} = x^2yxx^{-1} = x(xy)x^2
\]

\[
= xyxx^2 = (y^4)x xx^2 = y^16x^4 = y^7x^4.
\]

Alternatively, \( x^4y^4 = (yx)^{-1}y^4 = yxy^3 = y^4x \). Setting \( y^7x^4 = y^4x \) and cancelling yields \( y^3 = x^{-3} \), so \( x^3y^3 = 1 \). Since \( Z(G) = \langle x^3, y^3 \rangle \) and \( x^3 \) has order 3, we conclude that \( |Z(G)| = 3 \) and so \( |G| = 27 \). As a consequence, \( G \) is isomorphic to the subgroup \( \langle u, v \rangle \) of \( S_9 \) given above. Furthermore, since \( G = \{ x^ry^s : 0 \leq r, s \leq 8 \} \) and \( y^3 = x^{-3} \), we can further say that \( G = \{ x^ry^s : 0 \leq r \leq 8, 0 \leq s \leq 2 \} \). Therefore, every element of \( G \) has a unique expression in the form \( x^ry^s \) with \( 0 \leq r \leq 8, 0 \leq s \leq 2 \). Moreover, our conjugacy relations yield a formula for multiplication in \( G \). We see that

\[
(x^ry^s)(x^ny^m) = x^r(\begin{pmatrix} y^s \end{pmatrix}x^n\begin{pmatrix} y^{-s} \end{pmatrix})(\begin{pmatrix} y^s \end{pmatrix}y^n) = x^r(\begin{pmatrix} y^s \end{pmatrix}xy^{-s})^n y^{s+n}
\]

\[
= x^r\begin{pmatrix} x^s \end{pmatrix}^n y^{s+n}
\]

\[
= x^{r+ns}y^{s+n}.
\]

While this isn’t extremely simple, it does completely determine the multiplication.

To help motivate our discussion of group extensions, we try to get a better idea of the structure of \( G \). The exact sequence \( 1 \to Z(G) \to G \to G/Z(G) \to 1 \) simplifies to \( 1 \to \mathbb{Z}_3 \to G \to \mathbb{Z}_3 \times \mathbb{Z}_3 \to 1 \). Thus, we completely understand the subgroup \( Z(G) \) and the quotient group \( G/Z(G) \). However, there are several groups containing a normal subgroup \( N \) isomorphic to \( \mathbb{Z}_3 \) and with quotient group \( G/N \) isomorphic to \( \mathbb{Z}_3 \times \mathbb{Z}_3 \). For example, \( \mathbb{Z}_{27} \), \( \mathbb{Z}_9 \times \mathbb{Z}_3 \), and \( \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \) are three other such groups. In fact, even specifying \( Z(G) \cong \mathbb{Z}_3 \) and \( G/Z(G) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \) is not enough; we give a second example below. More information than the subgroup \( N \) and quotient group \( G/N \) is needed to determine which group we have.

There is another way to realize \( G \) by working with a different normal subgroup. Let \( N = \langle x \rangle \). This is normal in \( G \) since \( yxy^{-1} = x^4 \). Let \( z = xy \). We have seen that \( z^3 = 1 \).
Furthermore, \( zxz^{-1} = xyxy^{-1}x^{-1} = x^4 \). We claim that \( G = \langle x, z : x^9 = z^3 = 1, zxz^{-1} = x^4 \rangle \). Let \( G' \) be this presented group. From the relations a short argument shows that \( |G'| \leq 27 \). However, we have a homomorphism from \( G' \) to \( G \) by sending \( x \) to \( x \) and \( z \) to \( xy \). The elements \( x \) and \( xy \) clearly generate \( G \); therefore, this homomorphism is surjective. Since \( |G| = 27 \), we conclude that \( |G'| = 27 \) and so \( G' \cong G \). Moreover, if \( H = \langle xy \rangle \), then we have \( G = NH \) and \( N \cap H = 1 \). Since \( N \) is normal in \( G \), this shows that \( G \) is the semidirect product of \( N \) and \( H \). Furthermore, defining \( \psi : H \to \text{Aut}(N) \) by \( \psi(h) \) is conjugation by \( h \), we obtain a group homomorphism. Since \( N \cong \mathbb{Z}_9 \), we have \( \text{Aut}(N) \cong \mathbb{Z}_9^* \cong \mathbb{Z}_6 \). The automorphism \( \psi(z) \) is the map with \( x \mapsto x^4 \). In the standard notation for \( \text{Aut}(\mathbb{Z}_9) \), we have \( \psi(z) = \sigma_4 \). This is an automorphism of order 3 since \( 4^3 \equiv 1 \mod 9 \). Consequently, the homomorphism \( \psi \) is injective. As we will see, the semidirect product is determined by the subgroups \( N \) and \( H \) together with the map \( \psi \). Since \( N \) and \( H \) are Abelian, the map \( \psi \) must be nontrivial in order to obtain a non-Abelian group.

In case one wonders if the properties \( Z(G) \cong \mathbb{Z}_3 \) and \( G/Z(G) \cong \mathbb{Z}_9 \times \mathbb{Z}_3 \) together with \( G \) non-Abelian are enough to determine \( G \), we give a second example. This example is closely related to the quaternion group \( Q_8 \). Let \( \omega \) be a primitive 3rd root of unity; e.g., \( \omega = \exp(2\pi i/3) \). Let \( Q_{27} \) be the set of symbols \( \{\omega^n i^m j^n : 0 \leq n, m, p \leq 2\} \), subject to the relations \( i^3 = j^3 = \omega \) and \( ji = \omega ij \) along with \( \omega \in Z(Q_{27}) \). Then \( |Q_{27}| = 27 \). Calculations like those we have done for the quaternion group show \( Z(Q_{27}) = \langle \omega \rangle \cong \mathbb{Z}_3 \). Moreover, \( Q_{27}/Z(Q_{27}) \) is generated by \( \bar{i} \) and \( \bar{j} \), and since \( i^3 = j^3 = \omega \), we have \( \bar{i}^3 = \bar{j}^3 = 1 \). Moreover, \( \bar{i} \bar{j} = \bar{\omega} \bar{i} \bar{j} = \bar{j} \bar{i} \). Thus, \( Q_{27}/Z(Q_{27}) \) is Abelian. The group \( Q_{27}/Z(Q_{27}) \) must then be isomorphic to \( \mathbb{Z}_3 \times \mathbb{Z}_3 \). To finish this note we state without the fairly easy proof that \( Q_{27} \) has the presentation \( \langle x, y : x^9 = y^9 = 1, x^3 = y^3, yx = x^4y \rangle \). This presentation looks somewhat similar to that of the group \( G \) we studied earlier. In fact, these relations are all ones satisfied by that group except that here \( x^3 = y^3 \) while for \( G \) we had \( x^3 = y^{-3} \). In fact, an alternate presentation of \( G \) is \( \langle x, y : x^9 = y^9 = 1, x^3 = y^{-3}, yx = x^4y \rangle \), so the only difference is the negative exponent in the third relation. However, while \( G \) can be described as a semidirect product, \( Q_{27} \) cannot be so described; one can show that every nontrivial subgroup of \( Q_{27} \) contains \( \omega \); therefore, there do not exist two nontrivial subgroups \( N \) and \( H \) of \( Q_{27} \) with \( N \cap H = 1 \), a requirement to have a semidirect product.