Some Examples of Wallpaper Groups

In this note we discuss some symmetry groups as nice examples of group extensions. We begin by a short general discussion of symmetry groups. An isometry of $\mathbb{R}^n$ is a distance-preserving bijection of $\mathbb{R}^n$. For examples of isometries, translations by a vector, rotations about a point, and reflections about a hyperplane are all isometries. Let $\text{Isom}(\mathbb{R}^n)$ be the set of all isometries of $\mathbb{R}^n$. This is a group under composition of functions.

If $S$ is a subset of $\mathbb{R}^n$, then the symmetry group of $S$ is defined to be

$$\text{Sym}(S) = \{ \varphi \in \text{Isom}(\mathbb{R}^n) : \varphi(S) = S \}.$$ 

In other words, the symmetry group of $S$ is the set of all isometries of $\mathbb{R}^n$ that stabilize the set $S$.

The group structure of $\text{Isom}(\mathbb{R}^n)$ is fairly nice. It has a subgroup $T$ consisting of all translations. We will denote by $\tau_v$ the translation by a vector $v$. Thus, $\tau_v(x) = x + v$ for all $x \in \mathbb{R}^n$. The group $T$ is isomorphic to $\mathbb{R}^n$ under the map $v \mapsto \tau_v$. It is a normal subgroup, since if $\varphi \in \text{Isom}(\mathbb{R}^n)$, then $\varphi \tau_v \varphi^{-1} = \tau_{\varphi(v)}$. There is another important subgroup, the subgroup of all linear isometries. These are the isometries which are also linear transformations of the vector space $\mathbb{R}^n$. Suppose that $\varphi$ is a linear isometry. By choosing a basis, we may represent $\varphi$ with a matrix $A$. The condition that $\varphi$ is distance preserving says that $\|Av - Aw\| = \|v - w\|$ for all $v, w \in \mathbb{R}^n$. Thus, by replacing $v - w$ by $x$, we see that $\|Ax\| = \|x\|$. If we view vectors as column matrices, then $\|x\|^2 = x \cdot x = x^T x$. Thus, the condition $\|Ax\| = \|x\|$ can be written as $(Ax)^T(Ax)$, or $x^T(A^T A)x = x^T x$. A calculation via bases will show that if this equation holds for all $x$, then $A^T A = I$. Therefore, the group of linear isometries is isomorphic to the matrix group

$$O_n(\mathbb{R}) = \{ A \in \text{Gl}_n(\mathbb{R}) : A^T A = I \}.$$ 

While we won’t prove it here, the subgroup of linear isometries is the same as the subgroup of isometries that fix the origin. Thus, if $\varphi$ is an arbitrary isometry and $\varphi(0) = v$, then $\tau_{-v} \varphi$ sends 0 to 0, so it is a linear isometry. Thus, every isometry is the composition of a linear isometry and a translation. We will use the notation $(A, v)$ to represent the isometry that sends an arbitrary vector $x$ to $Ax + v$, where $v \in \mathbb{R}^n$ and $A \in O_n(\mathbb{R})$. With this notation, we have the multiplication formula


We have a split exact sequence

$$0 \to \mathbb{R}^n \xrightarrow{i} \text{Isom}(\mathbb{R}^n) \xrightarrow{\pi} O_n(\mathbb{R}) \to 0$$
where \( i(v) = (I, v) \) and \( \pi(A, v) = A \). Furthermore, the splitting map \( O_n(\mathbb{R} \to \text{Isom}(\mathbb{R}^n)) \) is given by \( A \mapsto (A, 0) \). We point out that the action arising from this extension is the “natural” action of \( O_n(\mathbb{R}) \) on \( \mathbb{R}^n \) given by \( A \cdot v = Av \), usual matrix multiplication.

We now focus on \( \mathbb{R} \). To help discuss symmetry groups in the plane, we make the following note about \( O_2(\mathbb{R}) \). If \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O_2(\mathbb{R}) \), then \( A^T A = I \). Writing this out, we see that the two columns of \( A \) form an orthonormal basis for \( \mathbb{R}^2 \). Because the first column then has length 1, there is a unique angle \( \theta \) with \( (a, c) = (\cos \theta, \sin \theta) \). Next, since the second column is orthogonal to the first and has length 1, there are two possibilities for \( (b, d) \) (draw a picture to see this); one choice is \((- \sin \theta, \cos \theta)\) and the other choice is \((\sin \theta, -\cos \theta)\). Therefore, either

\[
A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]
or

\[
A = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} = \begin{pmatrix} \cos \theta/2 & -\sin \theta/2 \\ \sin \theta/2 & \cos \theta/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos \theta/2 & \sin \theta/2 \\ -\sin \theta/2 & \cos \theta/2 \end{pmatrix}.
\]

In the first case \( A \) represents a rotation about an angle \( \theta \). In the second case, \( A \) represents a reflection about a line making an angle of \( \theta/2 \) with the \( x \)-axis. Therefore, every element of \( O_2(\mathbb{R}) \) represents either a rotation or a reflection. One way to tell whether a linear isometry of the plane is a rotation or a reflection is by using determinants; if \( \det(A) = 1 \), then \( A \) is a rotation, and if \( \det(A) = -1 \), then \( A \) is a reflection.

Let \( G = \text{Sym}(W) \) be a symmetry group in the plane, so \( G \) is a subgroup of \( \text{Isom}(\mathbb{R}^2) \). If \( T = G \cap T \), then \( T \) is a normal subgroup of \( T \), being the kernel of \( \pi|_G \). The quotient group \( G/T \) is then isomorphic to a subgroup of \( O_2(\mathbb{R}) \); we will write \( G_0 \) for this group. We have a short exact sequence

\[
0 \to T \to G \to G_0 \to 0.
\]

We will consider in this note four examples of \textit{wallpaper groups}; these are symmetry groups whose translation subgroup \( T \) is isomorphic to \( \mathbb{Z} \times \mathbb{Z} \). What this means is that the set \( W \) has translational symmetry in two independent directions, and that every translation is a linear combination of two such symmetries. The four groups we consider are \( p2 \), \( pg \), \( p4m \), and \( p4g \). These names are standard names given to these groups by crystallographers. If you are curious to know the meaning of these names, consult a book which discusses wallpaper groups.

To help understand the group theoretic properties of these symmetry groups, we point out without proof that \( G_0 \) is a finite group for any wallpaper group; this will be evident in our examples. If \( G_0 \) is isomorphic to a subgroup of \( G \), then \( G \) necessarily will be a semidirect product of \( T \) and \( G_0 \) since the condition \( T \cap G_0 = 1 \) follows since \( G_0 \) is finite and \( T \) has no nonzero element of finite order.
Consider the symmetry group of the following drawing of Escher.

Look carefully at the picture to see the grid Escher drew. Let \( t_1 \) be the smallest translation vector along the grid lines with positive slope, and let \( t_2 \) be the smallest translation vector along the lines of negative slope. The translation subgroup \( T \) of the symmetry group \( G \) is 
\[
T = \{(I, nt_1 + mt_2) : n, m \in \mathbb{Z}\} \cong \mathbb{Z} \times \mathbb{Z}.
\]
By looking carefully at the picture, we see that there is a \( 180^\circ \) rotational symmetry \( r \), and that \( G = \langle (I, t_1), (I, t_2), (r, 0) \rangle \). The group \( G_0 \) is \( \langle r \rangle \cong \mathbb{Z}_2 \). We then have the group extension
\[
\mathbb{Z} \times \mathbb{Z} \to G \to \mathbb{Z}_2 \to 0.
\]
Viewing \( \mathbb{Z}_2 \cong \langle r \rangle \), the action on \( T \) is given by \( r(t) = -t \), since \( r \) is rotation by \( 180^\circ \). The group \( G \) is a semidirect product of \( T \) by \( G_0 \), since \( G_0 \) is a subgroup of \( G \). In terms of generators and relations,
\[
G = \langle \tau_1, \tau_2, r : \tau_1 \tau_2 = \tau_2 \tau_1, r^2 = 1, r \tau_1 r^{-1} = \tau_1^{-1}, r \tau_2 r^{-1} = \tau_2^{-1} \rangle.
\]

Next, consider the following picture of Escher.
By a careful investigation to this picture, we see that the symmetry group $G$ is generated by $(I, t_1)$, $(I, t_2)$, and $g = (f, \frac{1}{2}t_2)$, where $f$ is a reflection about a vertical line, $t_1$ is the smallest horizontal translation vector, and $t_2$ is the smallest vertical translation vector. In this case $G_0 \cong \langle f \rangle \cong \mathbb{Z}_2$. Therefore, $G$ is given by a group extension

$$0 \to \mathbb{Z} \times \mathbb{Z} \to G \to \mathbb{Z}_2 \to 0$$

as in the previous example. However, two things are different. First, the action is different, for $f(t_1) = -t_1$ and $f(t_2) = t_2$. Second, $G$ is not the semidirect product of these two groups since $G_0$ is not a subgroup of $G$ as the figure does not have any reflections. In terms of generators and relations,

$$G = \langle \tau_1, \tau_2, g : \tau_1\tau_2 = \tau_2\tau_1, g^2 = \tau_2, g\tau_1g^{-1} = \tau_1^{-1}, g\tau_2g^{-1} = \tau_2 \rangle.$$

\textbf{p4m}

The next group we consider is the symmetry group of the following drawing of Escher.
If \( t_1 \) is the smallest horizontal translation vector and \( t_2 \) is the smallest vertical translation vector, if \( r \) is a \( 90^\circ \) rotation about the center and \( f \) is the reflection about the \( x \)-axis, then

\[
G = \langle (I, t_1), (I, t_2), (r, 0), (f, 0) \rangle.
\]

The group \( G_0 \) is then \( \langle r, f \rangle \cong D_4 \). This group acts on \( \mathbb{Z} \times \mathbb{Z} \) by

\[
\begin{align*}
  r(t_1) &= t_2, & r(t_2) &= -t_1, \\
  f(t_1) &= t_1, & f(t_2) &= -t_2.
\end{align*}
\]

The corresponding group extension is

\[
0 \rightarrow \mathbb{Z} \times \mathbb{Z} \rightarrow G \rightarrow D_4 \rightarrow 0,
\]

and \( G \) is the semidirect product of \( T \) and \( D_4 \) since \( D_4 \) is a subgroup of \( G \). In terms of generators and relations,

\[
G = \langle \tau_1, \tau_2, r, f : \tau_1 \tau_2 = \tau_2 \tau_1, r^4 = f^2 = 1, frf = r^{-1}, \\
  r\tau_1 r^{-1} = \tau_2, r\tau_2 r^{-1} = \tau_1^{-1}, f\tau_1 f = \tau_1, f\tau_2 f = \tau_2^{-1} \rangle.
\]

\textbf{p4g}

The final group we consider is the symmetry group of the following drawing of Escher.

In this picture, if we consider the origin to be the center of the picture, where four wings touch, then the translation subgroup is generated by \( (I, t_1) \) and \( (I, t_2) \), where \( t_1 \) is the smallest horizontal translation vector and \( t_2 \) is the smallest vertical translation. Moreover, if \( r \) is a rotation by \( 90^\circ \) about the origin and \( f \) is the reflection about the line spanned by \( t_1 \), then

\[
G = \left\langle (I, t_1), (I, t_2), (r, 0), (f, \frac{1}{2}(t_1 + t_2)) \right\rangle.
\]
The group $G_0$ in this case, as in the previous example, is $\langle r, f \rangle \cong D_4$, and our group extension is

$$0 \rightarrow \mathbb{Z} \times \mathbb{Z} \rightarrow G \rightarrow D_4 \rightarrow 0.$$ 

Furthermore, the action is the same as for the previous example. However, this extension is not the semidirect product. For example, $f$ is not an element of $G$, unlike the previous example. In our terminology of producing a cocycle $c$ from the group extension, we have a map $l : D_4 \rightarrow G$ for which $l(r^i) = (r^i, 0)$ and $l(fr^i) = (fr^i, v)$; we abbreviate $v = \frac{1}{2}(t_1 + t_2)$. If we wish to compute the cocycle $c$ corresponding to this choice, we have

$$c(f, f) = l(f)l(f)(id)^{-1} = (f, v)(f, v) = (I, v + f(v)) = (I, t_1).$$

Viewing $\mathbb{Z} \times \mathbb{Z}$ as a subgroup of $G$, we write $c(f, f) = t_1$. While this does not by itself prove that $c$ is not a coboundary, it does give evidence to this fact. This final group has the presentation

$$G = \langle \tau_1, \tau_2, r, g : \tau_2 \tau_1 = \tau_1 \tau_2, r^4 = 1, g^2 = \tau_1 \tau_2, grg^{-1} = r \tau_2, r \tau_1 r^{-1} = \tau_2, r \tau_2 r^{-1} = \tau_1^{-1}, g \tau_1 g^{-1} = \tau_1, g \tau_2 g^{-1} = \tau_2^{-1} \rangle.$$ 

One can see that the two final groups are different by considering the pattern of reflections and rotations.

View the origin as the bottom left black dot, the vector $t_1$ going from the bottom left to the bottom right black dot, and $t_2$ as the corresponding vertical vector. The lines given are then reflection lines in the two pictures. The group $D_4$ is not a subgroup of the final group because we do not have a reflection $f$ such that $frf = r^{-1}$. Any reflection composed with the rotation $r$ results in an isometry that moves the origin, since no reflection of the figure preserves the origin.